ELSEVIER

Physica A 274 (1999) 182–189

PHYSICA A

www.elsevier.com/locate/physa

Application of statistical mechanics to collective motion in biology

Tamás Vicsek[a,*], András Czirók[a], Illés J. Farkas[a], Dirk Helbing[b]

[a]*Department of Biological Physics, Eötvös University, Pázmány Péter Sétány 1A, H-1117 Budapest, Hungary*

[b]*II. Institute of Theoretical Physics, University of Stuttgart, Pfaffenwaldring 57/III, 70550 Stuttgart, Germany*

Abstract

Our goal is to describe the collective motion of organisms in the presence of fluctuations. Therefore, we discuss biologically inspired, inherently non-equilibrium models consisting of self-propelled particles. In our models the particles corresponding to organisms locally interact with their neighbours according to simple rules depending on the particular situation considered. Numerical simulations indicate the existence of new types of transitions. Depending on the control parameters both disordered and long-range ordered phases can be observed. In particular, we demonstrate that (i) there is a transition from disordered to ordered motion at a finite noise level even in one dimension and (ii) particles segregate into lanes or jam into a crystalline structure in a model of pedestrians.

1. Introduction

The collective motion of organisms (birds, for example), is a fascinating phenomenon many times capturing our eyes when we observe our natural environment. In addition to the aesthetic aspects of collective motion, it has some applied aspects as well: a better understanding of the swimming patterns of large schools of fish can be useful in the context of large scale fishing strategies. Our interest is also motivated by the recent developments in areas related to statistical physics. During the last 15 years or so there has been an increasing interest in the studies of far-from-equilibrium systems typical in our natural and social environment. Concepts originated from the physics of phase transitions in equilibrium systems [1–3] such as collective behaviour, scale invariance and renormalization have been shown to be useful in the understanding of

* Corresponding author.
E-mail address: h845vic@ella.hu (T. Vicsek)

0378-4371/99/$ - see front matter
PII: S0378-4371(99)00317-9

various non-equilibrium systems as well. Simple algorithmic models have been helpful in the extraction of the basic properties of various far-from-equilibrium phenomena, like diffusion limited growth [4,5], self-organized criticality [6] or surface roughening [7]. Motion and related transport phenomena represent a further characteristic aspect of non-equilibrium processes [8], including traffic models [9,10], thermal ratchets [11] or driven granular materials [12,13].

Self-propulsion is an essential feature of most living systems. In addition, the motion of the organisms is usually controlled by interactions with other organisms in their neighbourhood and randomness plays an important role as well. In Ref. [14] a simple model of self propelled particles (SPP) was introduced capturing these features with a view toward modelling the collective motion [15] of large groups of organisms such as schools of fish, herds of quadrupeds, flocks of birds, or groups of migrating bacteria, correlated motion of ants [16] or pedestrians [17,18].

It turns out that modelling collective motion leads to interesting specific results inscr2 with the relevant dimensions (from 1–3 [19,14,20,21]). Here we discuss a one-dimesional and a quasi-one-dimensional version of the related models.

2. Ordered motion for finite noise in 1d

Since in 1d the particles cannot get around each other, some of the important features of the dynamics present in higher dimensions are lost. On the other hand, motion in 1d implies new interesting aspects (groups of the particles have to be able to change their direction for the opposite in an organized manner) and the algorithms used for higher dimensions should be modified to take into account the specific crowding effects typical for 1d (the particles can slow down before changing direction and dense regions may be built up of momentarily oppositely moving particles).

In a way the system we study can be considered as a model of organisms (people, for example), moving in a narrow channel. Imagine that a fire alarm goes on, the tunnel is dark, smoky, everyone is extremely excited. People are both trying to follow the others (to escape together) and behave in an erratic manner (due to smoke and excitement). Will they escape (move in a selected direction in an orderly manner) or become trapped (keep moving back and forth)? One way to study this and related applied questions is carrying out simulations.

Thus, we consider N off-lattice particles along a line of length L. The particles are characterized by their coordinate x_i and dimensionless velocity u_i updated as

$$x_i(t+\Delta t) = x_i(t) + v_0 u_i(t)\Delta t\,, \tag{1}$$

$$u_i(t+\Delta t) = G(\langle u(t)\rangle_{S(i)}) + \xi_i\,, \tag{2}$$

where ξ_i is a random number drawn with a uniform probability from the interval $[-\eta/2, \eta/2]$. The local average velocity $\langle u\rangle_{S(i)}$ for the ith particle is calculated over the particles located in the interval $[x_i-\Delta, x_i+\Delta]$, where we fix $\Delta = 1$. The function G tends to set the velocity in average to a prescribed value v_0: $G(u) > u$ for $u < 1$

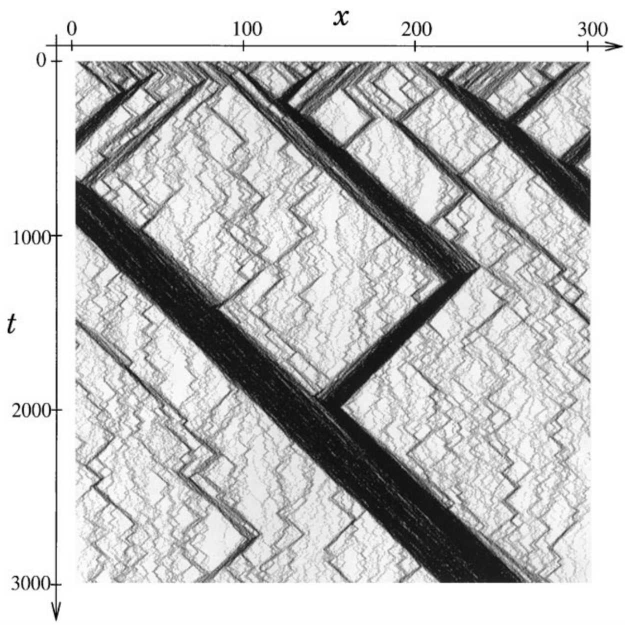

Fig. 1. The first 3000 time steps of the 1d SPP model [$L = 300$, $N = 600$, $\eta = 2.0$ (a) and $\eta = 4.0$ (b)]. The darker grey scale represents higher particle density.

and $G(u) < u$ for $u > 1$. In the numerical simulations [13] one of the simplest choices for G was implemented as

$$G(u) = \begin{cases} (u+1)/2 & \text{for } u > 0\,, \\ (u-1)/2 & \text{for } u < 0 \end{cases} \tag{3}$$

and random initial and periodic boundary conditions were applied.

In Fig. 1 we show the time evolution of the model for $\eta = 2.0$. In a short time the system reaches an ordered state, characterized by a spontaneous broken symmetry and clustering of the particles. In contrast, for $\eta = 4.0$ the system remains in a disordered state. Ordering for finite noise in 1d is a surprising phenomenon and is entirely due to the specific non-equilibrium nature of our system.

To understand the phase transitions observed in the various SPP models, efforts has been made to set up a consistent continuum theory in terms of v and ρ, representing the coarse-grained velocity and density fields, respectively. The first approach [22,23] has been made by Toner and Tu for $d > 1$ who investigated the following set of equations

$$\partial_t v + (v\nabla)v = \alpha v - \beta|v|^2 v - \nabla P + D_L \nabla(\nabla v) + D_1 \nabla^2 v + D_2 (v\nabla)^2 v + \xi$$

$$\partial_t \rho + \nabla(\rho v) = 0\,, \tag{4}$$

where the $\alpha, \beta > 0$ terms make v have a non-zero magnitude, $D_{L,1,2}$ are diffusion constants and f is an uncorrelated Gaussian random noise. The pressure P depends on the local density only, as given by the expansion

$$P(\rho) = \sum_n \sigma_n(\rho - \rho_0)^n . \tag{5}$$

The non-intuitive terms (4th and 6th in the RHS) in Eq. (4) were generated by the renormalization process.

Tu and Toner were able to treat the problem analytically and show the existence of an ordered phase in 2D, and also extracted exponents characterizing the density–density correlation function. They showed that the upper critical dimension for their model is 4 and the theory does not allow an ordered phase in 1d.

However, as we have shown, there exist SPP systems in one dimension which exhibit an ordered phase for low noise level. This finding motivated us to construct an alternative continuum model for 1d [19].

$$\partial_t u = f(u) + \mu^2 \partial_x^2 u + \alpha \frac{(\partial_x u)(\partial_x \rho)}{\rho} + \zeta , \tag{6}$$

$$\partial_t \rho = -v_0 \partial_x(\rho u) + D\partial_x^2 \rho , \tag{7}$$

where $u(x,t)$ is the coarse-grained dimensionless velocity field, $f(u)$ is an antisymmetric function with $f(u) > 0$ for $0 < u < 1$ and $f(u) < 0$ for $u > 1$, $\overline{\zeta} = 0$, and $\overline{\zeta^2} = \sigma^2/\rho\tau^2$. These equations are different both from the equilibrium field theories and from the nonequilibrium system defined through Eq. (4). The main difference comes from the nature of the coupling term $(\partial_x u)(\partial_x \rho)/\rho$. This term can be derived, but here we present a plausible interpretation of its origin.

The main point is that one would like to have a term in the equation for the velocity which would result in the slowing down (and eventually in the "turning back") of the particles under the influence of a larger number of particles moving oppositely. When two groups of particles move in the opposite direction, the density locally increases and the velocity decreases at the point they meet. Let us consider a particular case, when particles move from left to right and the velocity is locally decreasing while the density is increasing as x increases (particles are moving towards a "wall" formed between two oppositely moving groups). The term $(\partial_x u)$ is less, the term $(\partial_x \rho)$ is larger than zero in this case. Together they have a negative sign resulting in the slowing down of the local velocity. This is a consequence of the fact that there are more slower particles (in a given neighbourhood) in the forward direction than faster particles coming from behind, so the average action (a particle is trying to take the average velocity of its neighbours and simultaneously move with a preferred velocity) experienced by a particle in the point x slows it down. Thus, the $(\partial_x u)(\partial_x \rho)/\rho$ term does what we expect from it.

Given the equation, and the picture of domain walls formed between oppositely moving groups of particles we can use the following simple arguments to interpret long range ordering.

A small domain of (left moving) particles moving opposite to the surrounding (right moving) ones is *bound to interact* with more and more right moving particles and, as a result, the domain wall assumes a specific structure which is characterized by a buildup of the right moving particles on its left side, while no more than the originally present left moving particles remain on the right-hand side of the domain wall. This process "transforms" non-local information (the size of the corresponding domains) into a *local asymmetry of the particle density* which, through the instability of the walls [19], results in a leftward motion of the domain wall, and consequently, eliminates the smaller domain.

This can be demonstrated schematically as

$$\begin{array}{c} >>>>>>>>>>><<<<<<<<>>>>>>>>>>>>> \\ \qquad\qquad A \qquad\qquad\qquad B \qquad\qquad \end{array}$$

where by $>(<)$ we denoted the right (left) moving particles. In contrast to the Ising model the A and B walls are very different and have non-equivalent properties. In this situation the B wall will break into a B_1 and B_2, moving in opposite directions, B_1 moving to the left and B_2 moving to the right, leaving the area $B_1 - B_2$ behind, which is depleted of particles.

$$\begin{array}{c} >>>>>>>>>>><<<<< \qquad >>>>>>>>>> \\ \qquad\qquad A \qquad\qquad B_1 \quad B_2 \qquad\qquad \end{array}$$

At the A boundary the two type of particles slow down, while, due to the instability shown to be present in the system, the wall itself moves in a certain direction, most probably to the right. Even in the other, extremely rare case (i.e., when the A wall moves to the left), an elimination of the smaller domain $(A - B_1)$ takes place since the velocity of the domain wall A is smaller than the velocity of the particles in the "bulk" and at B_1 where the local average velocity is the same as the preferred velocity of the particles. Thus, the particles tend to accumulate at the domain wall A, which again, through local interactions leads to the elimination of the domain $A - B_1$.

3. Simulations of pedestrians in a corridor

In our studies of collective phenomena in systems of moving organisms we also considered humans walking in opposite directions along a corridor. Here we assumed that half of the people tended to go from left to right (and vice versa) with a constant velocity, unless they encountered another person. The interaction potential had a hard core (corresponding to the physical size of a pedestrian) and was assumed to decay as a power law within a cutoff radius of about twice of the hard core. All this can be represented in the form of equations of motion which we solved numerically [24].

In addition to the interest in the properties of systems involving walkers [17], there are several further motivations to study such a model. A system of light (rising) and heavy (sinking) particles in a vertical column of fluid or a system of oppositely charged colloidal particles in an electric field represent potential applications of our model. In fact, the system we study is a generalization to the continuum case of the two-species driven lattice gas proposed recently [25], with a number of relevant modifications arising from the adaptation to the off-lattice case. In a wider context, these models can be considered as simplified paradigms of systems consisting of entities with opposing interests (drives). In the present work, we consider the behaviour of a limited number of particles in a confined geometry, and our results are primarily valid for this "mesoscopic" situation. In the quickly growing literature on mesoscopic systems there are many examples of the potential practical relevance of phenomena occurring in various models for finite sizes [26–28].

We started our simulations with N (typically between 24 and 72) particles which were randomly distributed in the corridor without allowing overlaps. For half of the particles a driving into the $(-1,0)$ direction, and for the other half a driving into the $(1,0)$ direction was assigned. Integrating equation using periodic boundary conditions produced the following results: For small noise amplitudes η and moderate densities, our simulations lead to the formation of coherently moving linear structures (just as if the particles moved along traffic lanes). For relatively large N, jamming occurs, depending on the respective initial condition. Nevertheless, lane formation, if at all, happens relatively quickly and is very dominant and robust in our model (in contrast with simpler lattice gas versions we have tested).

The mechanism of lane formation can be understood as follows: Particles moving against the stream or in areas of mixed directions of motion will have frequent and strong interactions. In each interaction, the encountering particles move a little aside in order to pass each other. This sidewards movement tends to separate oppositely moving particles. Moreover, once the particles move in uniform lanes, they will have very rare and weak interactions. Hence, the tendency to break up existing lanes is negligible, when the fluctuations are small. Furthermore, the most stable configuration corresponds to a state with a minimal interaction rate [29].

Whereas lane formation was also observed in previous studies of related models with *deterministic dynamics only* [30], in the present, more realistic model we have discovered a surprising phenomenon when we increased the noise amplitude. If the fluctuations were large enough, they were able to prevent lane formation or even to destroy previously existing lanes, eventually causing a mutual blocking of the opposite directions of motion for given, sufficiently high densities. In this blocked state, the particles were arranged in a hexagonal lattice structure, very much like in a crystal. In particular, there are densities, at which we observe *bistability*. That is, we find lanes at small noise amplitudes, but in a crystallized state, if the noise amplitude is large.

Thus, one can drive the system from a "fluid" state with lanes of uniform directions of motion to a "frozen" state just by increasing the fluctuations. We call this transition "freezing by heating" [24]. Although this transition exists for the simplest version of the model, it is more pronounced if the gradients of the repulsive potentials are multiplied with factors

$$\{\lambda + (1-\lambda)[1+\cos(\varphi_i)]/2\}\,, \tag{8}$$

where φ_i denotes the angle between the direction of motion e_i of particle i and the direction of the object exerting the repulsive force. The parameter λ reflects the relative influence of forces "from behind", in order to mimic that the interactions may depend on the (relative) velocities of the particles, a situation quite common in realistic driven systems.

Why is freezing by heating new? Some glasses may crystallize when slowly heated. However, this is a well understood phenomenon: the amorphous state is metastable for those temperatures, and crystallization means an approach to the more stable state with smaller free energy (smaller internal energy). In our case, crystallization is achieved by spontaneously driving the system with the help of noise uphill towards higher internal energy. The system would like to maximize its efficiency [29], but instead it ends up with minimal efficiency due to noise-induced crystallization. The role of "temperature" or noise here is to destroy the energetically more favourable fluid state, which inevitably leads to jamming. Crystallization occurs in this jammed phase, because it yields the densest packing of particles. The corresponding transition seems to be related to the off-lattice nature of our model and is different from those reported for driven diffusive systems on a lattice. It should be noted that the transition we find is not sharp, which is in part a consequence of the mesoscopic nature of the phenomenon.

We would like to point out that "freezing by heating" is likely to be relevant to situations involving pedestrians under extreme conditions (panics). Imagine a very smoky situation, caused by a fire, in which people do not know which is the right way to escape. When panicking, people will just try to get ahead, with a reduced tendency to follow a certain direction. Thus, fluctuations will be very large, which can lead to fatal blockings.

Acknowledgements

This work was supported by OTKA F019299, FKFP 0203/1997, and the DFG by the Heisenberg scholarship He 2789/1-1.

References

[1] S.-K. Ma, Statistical Mechanics, World Scientific, Singapore, 1985.
[2] S.-K. Ma, Modern Theory of Critical Phenomena, Benjamin, New York, 1976.
[3] H.E. Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford University Press, Oxford, 1971.
[4] T.A. Witten, L.M. Sander, Phys. Rev. Lett. 47 (1981) 1400.
[5] T. Vicsek, Fractal Growth Phenomena, 2nd Edition, World Scientific, Singapore, 1992.
[6] P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. Lett. 59 (1987) 381.
[7] A-L. Barabási, H.E. Stanley, Fractal Concepts in Surface Growth, Cambridge University Press, Cambridge, 1995.
[8] M. Schreckenberg, D.E. Wolf (Eds.), Traffic and Granular Flow '97, Springer, New York, 1998.
[9] See, e.g., D. Helbing, B.A. Huberman, Nature 396 (1998) 738.
[10] D. Helbing, A. Hennecke, M. Treiber, Phys. Rev. Lett. 82 (1999) 4360, and references therein.
[11] M.O. Magnasco, Phys. Rev. Lett. 71 (1993) 1477.
[12] Y.L. Duparcmeur, H.J. Herrmann, J.P. Troadec, J. Phys. (France) I 5 (1995) 1119.
[13] J. Hemmingsson, J. Phys. A 28 (1995) 4245.
[14] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Shochet, Phys. Rev. Lett. 75 (1995) 1226.

[15] C.W. Reynolds, Computer Graphics 21 (1987) 25.
[16] E.M. Rauch, M.M. Millonas, D.R. Chialvo, Phys. Lett. A 207 (1995) 185.
[17] D. Helbing, J. Keitsch, P. Molnar, Nature 387 (1997) 47.
[18] D. Helbing, F. Schweitzer, P. Molnar, Phys. Rev. E. 56 (1997) 2527.
[19] A. Czirók, A.-L. Barabási, T. Vicsek, Phys. Rev. Lett. 82 (1999) 209.
[20] A. Czirók, H.E. Stanley, T. Vicsek, Spontaneously ordered motion of self-propelled particles, J. Phys A 30 (1997) 1375.
[21] A. Czirók, M. Vicsek, T. Vicsek, Collective motion of organisms in three dimensions, Physica A 264 (1999) 299.
[22] J. Toner, Y. Tu, Phys. Rev. Lett. 75 (1995) 4326.
[23] J. Toner, Y. Tu, Phys. Rev. E 58 (1998) 4828.
[24] D. Helbing, I. Farkas, T. Vicsek, preprint cond-mat/9904326.
[25] B. Schmittmann, K. Hwang, R.K.P. Zia, Europhys. Lett. 19 (1992) 19.
[26] Y. Imry, Introduction to Mesoscopic Physics, Oxford University Press, Oxford, 1997.
[27] N.G. van Kampen, Stochastic Processes in Physics and Chemistry, North-Holland, Amsterdam, 1981.
[28] M-L. Tan, I. Goldhirsch, Phys. Rev. Lett. 81 (1998) 3022.
[29] D. Helbing, T. Vicsek, New J. Phys. 1 (1999) no. 13.
[30] D. Helbing, P. Molnár, Phys. Rev. E 51 (1995) 4282.