Fractal Growth Phenomena

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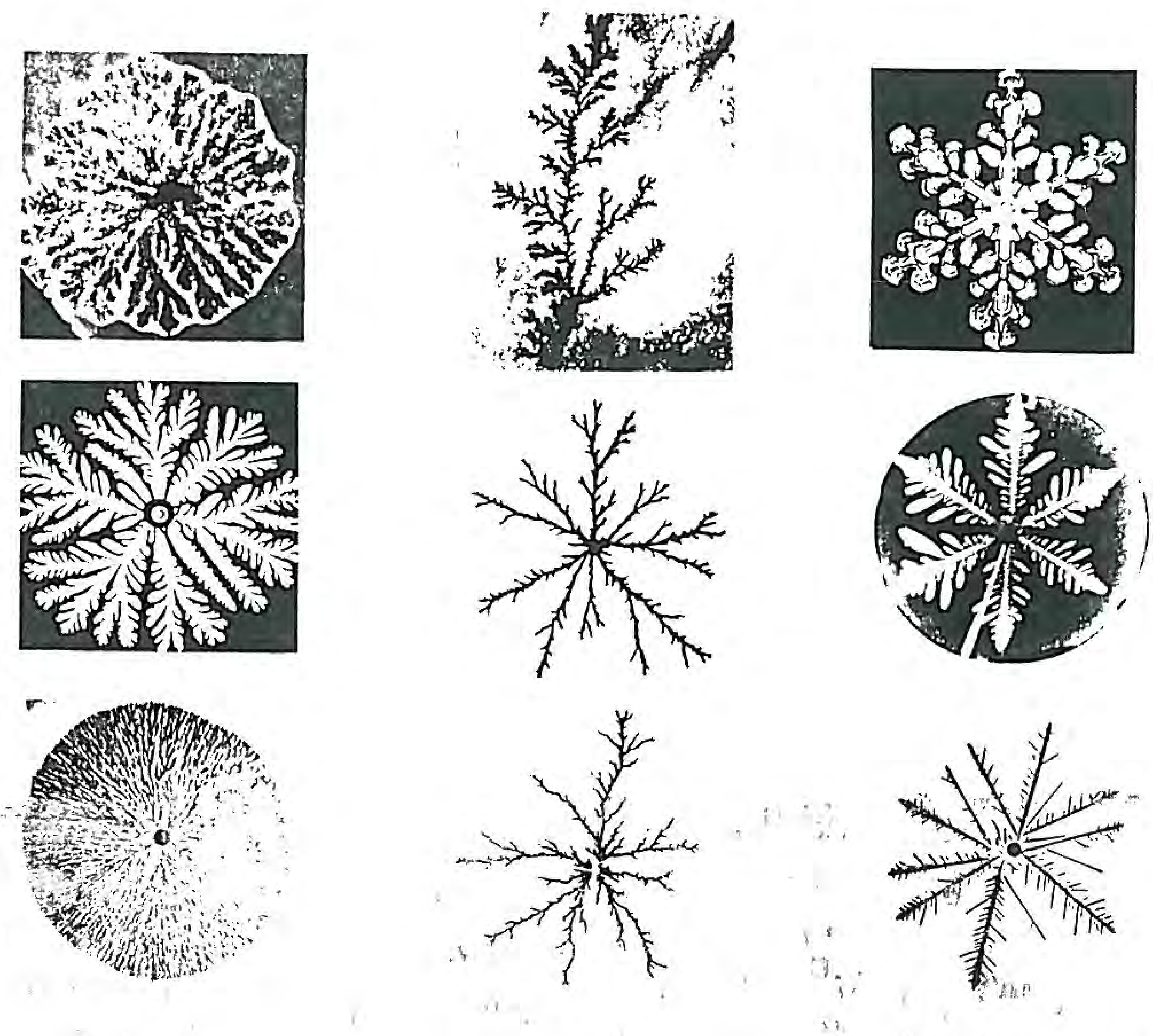

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$D = A \cdot b^{\alpha}$

$C(r) \sim A r^{\alpha}$ $r \to br$ $C(br) \sim D r^{\alpha}$

Scaling,
nem-analitikus, kritikus viselkedés, kritikus pont
univerzális kritikus exponensek

PART I.

FRACTALS

Chapter 3.

FRACTAL MEASURES

In the previous chapter such complex geometrical structures were discussed which could be interpreted in terms of a single fractal dimension. The present chapter is mainly concerned with the development of a formalism for the description of the situation when a singular distribution is defined on a fractal. As we shall see, the structure of fractals plays an essential role in the physical processes they are involved in and, as a result, one needs infinitely many dimension-type exponents to characterize these distributions (Mandelbrot 1974, Hentschel and Procaccia 1983, Benzi *et al* 1984, Frisch and Parisi 1984, Halsey *et al* 1986).

It is typical for a large class of physical phenomena that the behaviour of a system is determined by the spatial distribution of a scalar quantity, e.g. concentration, electric potential, probability, etc.. For simpler geometries this distribution function and its derivatives are relatively smooth, and they usually contain only a few (or none) singularities, where the word singular corresponds to a local power law behaviour of the function. (In other words, we call a function singular in the region surrounding point $\vec{x}$ if its local integral diverges or vanishes with a non-integer exponent when the integration size goes to zero. In the case of fractals the situation is quite different: a physical process involving a fractal may lead to a spatial distribution of the relevant quantities which possesses infinitely many singularities.

As an example, consider an isolated, charged object. If this object has sharp tips, the electric field around these tips becomes very large in accord with the behaviour of the solution of the Laplace equation for the potential. In the case of charging the branching fractals produced in the $k \to \infty$ limit of constructions shown in Fig. 2.1 or 2.5 one has infinite number of tips and corresponding singularities of the electric field. Moreover, tips being at different positions, in general have different local environments (configuration of the object in the region surrounding the given tip) which affect the strength of singularity associated with that position. There are further examples, some of which will be discussed later. At this point we only mention that additional physical phenomena leading to distributions with infinite number of singularities include, among many others, electric transport in percolation networks, viscous fingering, diffusion-limited aggregation, turbulence, chaotic motion, etc..

The interest in the properties of fractal measures has grown considerably in the last couple of years. A few reviews of the related results have been published very recently (Meakin 1987, Paladin and Vulpiani 1988, Tél 1988). When introducing certain concepts in the present chapter we follow the introductory paper by Tél (1988).

3.1. MULTIFRACTALITY

The above discussed time independent distributions defined on a fractal substrate are called *fractal measures*. In general, a fractal measure possesses an infinite number of singularities of infinitely many types. The term *"multifractality"* expresses the fact that points corresponding to a given type of singularity typically form a fractal subset whose dimension depends on the type of singularity. It is perhaps difficult to imagine how such an extremely complex distribution can appear as a result of simple physical processes. Let us, therefore, use a deterministic construction to demonstrate the mechanisms by which fractal measures can be generated.

We shall assume that the measure denoted by $\mu(\vec{x})$ is normalized so that its total amount on the fractal is equal to 1. Then, the fractal measure can be regarded as a probability distribution. Consider now a ***deterministic recursive (multiplicative) process*** generating a non-uniform fractal with varying weights or probabilities attributed to each rescaled part in a manner demonstrated in Fig. 3.1. In the first step the starting object (square) is replaced by its three smaller copies with corresponding reduction factors $1/r_1$, $1/r_2$ and $1/r_3$. In addition, the three newly created objects are given a weight factor (probability) denoted by P_1, P_2 and P_3. In the next step ($k = 2$) the same procedure is repeated for each of the squares, treating them as starting objects. To obtain a fractal measure this process has to be continued till $k \to \infty$ (Hentschel and Procaccia 1983).

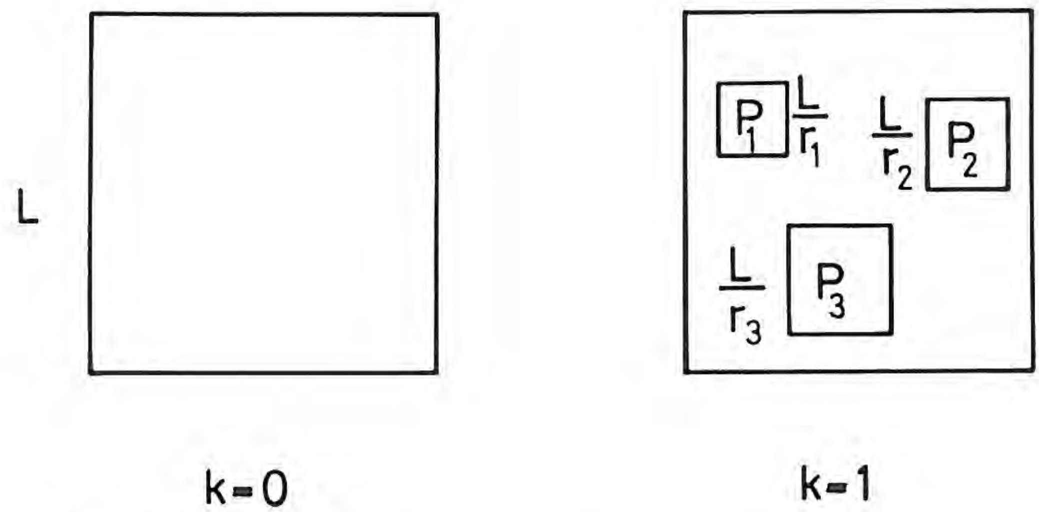

Figure 3.1. Construction of a fractal measure defined on a non-uniform fractal support embedded into two dimensions. The multifractal is obtained after infinitely many recursions.

It can be well seen from Fig. 3.1 that as the recursion advances a very complex situation emerges both concerning the size and the weight distribution of the objects. For example, if $P_1 > P_2$ and and $k \gg 1$, a square which appears as a result of going through reductions each time by the factor $1/r_1$, will have a much larger weight ($P_1^k \gg P_2^k$) than that being reduced each time by $1/r_2$.

In order to describe a fractal measure in a more quantitative way we imagine that a d dimensional hypercubic lattice with a lattice constant l is put on the fractal, and denote by p_i the stationary probability associated with the ith box of volume l^d, where

$$p_i = \int_{i\text{th box}} d\mu(\vec{x}) \tag{3.1}$$

with $\sum_i p_i = 1$. First we are interested in the behaviour of p_i as a function of the box size measured in units of the linear size L of the structure. As before this dimensionless unit will be denoted by $\epsilon = l/L$, and scaling of various quantities in the limit $\epsilon \to 0$ (corresponding to either $l \to 0$ or in the growing case to $L \to \infty$) will be studied. Trivially, for a homogeneous structure with a uniform distribution (density) $p_i(\epsilon) \sim \epsilon^d$. In the case of a uniform fractal ($r_i \equiv r$) of dimension D with a uniform distribution on it $p_i(\epsilon) \sim \epsilon^D$, since for a distribution with a uniform density p_i corresponds to the volume of the support in the ith box. (The term support is used for the fractal on which the measure is defined.)

In the more complex situation, when a non-uniform fractal with a distribution having infinitely many singularities is considered, we are led to assume a general form

$$p(\epsilon) \sim \epsilon^\alpha, \tag{3.2}$$

where $\epsilon \ll 1$ and α can take on a range of values depending on the given region of the measure. The non-integer exponent α corresponds to the strength of the local singularity of the measure and is sometimes also called crowding index or Hölder exponent. Although α depends on the actual position on the fractal, there are usually many boxes with the same index α. In general, the number of such boxes scales with ϵ as

$$N_\alpha(\epsilon) \sim \epsilon^{-f(\alpha)}, \tag{3.3}$$

where, in view of (2.4), $f(\alpha)$ is the fractal dimension of the subset of boxes characterized by the exponent α. The exponent α can take on values from the interval $[\alpha_\infty, \alpha_{-\infty}]$, and $f(\alpha)$ is usually a single humped function (see Example 3.1. in Section 3.3.) with a maximum

$$\max_{\alpha} f(\alpha) = D. \tag{3.4}$$

The $f(\alpha)$ spectrum of an ordinary uniform fractal is a single point on the $f - \alpha$ plane.

Thus, a typical fractal measure is assumed to be made of interwoven sets of singularities of strength α, each characterized by its own fractal dimension $f(\alpha)$. This fact is the reason for the name multifractal frequently used for such fractal measures. In order to determine $f(\alpha)$ for a given distribution, it is useful to introduce a few quantities which are more directly related to the observable properties of the measure. Then, relations among these quantities and $f(\alpha)$ make it possible to obtain a complete description of a fractal measure (Halsey *et al* 1986).

3.2. RELATIONS AMONG THE EXPONENTS

An important quantity which can be determined from the weights p_i is the sum over all boxes of the qth power of box probabilities

$$\chi_q(\epsilon) \equiv \sum_i p_i^q \tag{3.5}$$

for $-\infty < q < \infty$. This definition of χ is closely related to the Rényi entropies (Rényi 1970, Hentschel and Procaccia 1983). For $q = 0$ (3.5) gives $N(\epsilon)$, the number of boxes of size ϵ needed to cover the fractal support (the region, where $p_i \neq 0$). Therefore,

$$\chi_0(\epsilon) = N(\epsilon) \sim \epsilon^{-D}, \tag{3.6}$$

where D is the dimension of the support. Since the distribution is normalized, $\chi_1(\epsilon) = 1$.

Because of the complexity of multifractal distributions the scaling of $\chi_q(\epsilon)$ for $\epsilon \to 0$ generally depends on q in a non-trivial way

$$\chi_q(\epsilon) \sim \epsilon^{(q-1)D_q}, \tag{3.7}$$

where D_q is the so called *order q generalized dimension*. It should be noted, however, that in spite of their name the D_q-s are not fractal dimensions, as we shall see later. The factor $(q-1)$ was chosen to satisfy the relation $\chi_1(\epsilon) = 1$ automatically. Correspondingly, for all q we have $D_q > 0$. It can be shown that the D_q values monotonically decrease with growing q. From the comparison of (3.6) with (3.7) it follows that $D_0 = D$. In the simple case of uniform fractals with a uniform distribution all D_q-s are equal to D. The typical behaviour of D_q as a function of q can be seen from Example 3.1. in the next section.

The distribution of p_i-s is extremely inhomogeneous concerning both their values and the number of boxes with the same p_i. As a result, when $\epsilon \to 0$ the dominant contribution to the sum (3.5) comes from a subset of all possible boxes. This subset forms a fractal with a fractal dimension f_q depending on the actual value of q, thus

$$N_q(\epsilon) \sim \epsilon^{-f_q}, \tag{3.8}$$

where $N_q(\epsilon)$ is the number of boxes giving the essential contribution in (3.5). In addition, all of these boxes have the same $p_i = p_q$. We shall denote the singularity strength for boxes with probability p_q by α_q, i.e.,

$$p_q \sim \epsilon^{\alpha_q} \tag{3.9}$$

for $\epsilon \to 0$. The f_q and α_q spectra defined by (3.8) and (3.9) provide an alternative description of a fractal measure with regard to (3.2) and (3.3).

What is the origin of the fact that a given q selects a fractal subset of dimension f_q with a corresponding crowding index α_q? When $\epsilon \to 0$ the contribution of a box depends very sensitively both on q and α, in particular, increasing $q > 1$ or α a bit, results in a dramatic decrease of the contribution coming from the same box. On the other hand, in the same limit the number of boxes having a little different probability than the given box, is very much different. The main contribution comes from boxes for which the product $N(\alpha, \epsilon)p_i^q$ is maximal, where $N(\alpha, \epsilon)$ is the number of boxes with α. For all other values of $\alpha \neq \alpha_q$ the contribution of the corresponding boxes is negligible.

According to (3.6) and (3.8) f_0 is equal to the dimension of the supporting fractal. It can be easily shown that the dimension of a fractal is always larger than the dimension of any of its subsets (see rule e) in Section 2.2.), therefore, $f_0 > f_q$ for $q \neq 0$. f_q as a function of growing q first increases up to D then monotonically decreases. Since for increasing q the dominant contribution comes from boxes with larger probabilities p_q, α_q is a monotonically decreasing function. For a simple fractal $f_q = \alpha_q = D$.

There is a relation between D_q and the spectra f_q and α_q which can be easily obtained taking into account that

$$\chi_q(\epsilon) \simeq N_q(\epsilon)p_q^q, \tag{3.10}$$

since the essential contribution to $\chi_q(\epsilon)$ comes from boxes with p_q. Inserting (3.7), (3.8) and (3.9) into (3.10) we get

$\chi_q(\varepsilon) \sim \varepsilon^{(q-1)D_q}$
$N_q(\varepsilon) \sim \varepsilon^{-f_q}$
$p_q \sim \varepsilon^{\alpha_q}$

$$(q-1)D_q = q\alpha_q - f_q. \tag{3.11}$$

The above expression is consistent with the earlier observation that $D = D_0 = f_0$. Moreover, since f is finite we have $D_{\pm\infty} = \alpha_{\pm\infty}$.

Now it is possible to relate the generalized dimensions to the $f(\alpha)$ spectrum. First we express $\chi_q(\epsilon)$ for $\epsilon \to 0$ through $f(\alpha)$ using (3.2), (3.3) and (3.5)

$$\chi_q(\epsilon) \sim \int_{\alpha_\infty}^{\alpha_{-\infty}} \epsilon^{q\alpha' - f(\alpha')}\, d\alpha', \tag{3.12}$$

where α is a quasi-continuous variable. For $\epsilon \ll 1$ the integral will be dominated by the value of α which minimizes the exponent. This leads to the condition

$$\frac{df(\alpha)}{d\alpha}|_{\alpha_q} = q, \tag{3.13}$$

where α_q is the value of α for which $q\alpha - f(\alpha)$ is minimal. We see that (3.12) is consistent with (3.11) if, in addition to (3.13),

$$f_q = f(\alpha_q). \tag{3.14}$$

Thus, knowing $f(\alpha)$ we can find α_q from (3.13) and then f_q from (3.14) (Halsey *et al* 1986).

Alternatively, if D_q is given, one can calculate α_q from the relation

$$\alpha_q = \frac{d}{dq}[(q-1)D_q] \tag{3.15}$$

which can be obtained from (3.11) using (3.13) and (3.14). If we know α_q and D_q, f_q can be determined from (3.11) and finally, $f(\alpha)$ from (3.14). Therefore, the spectra D_q and $f(\alpha)$ represent equivalent descriptions of multifractals, since they are Legendre transforms of each other (Halsey *et al* 1986).

For the evaluation of experimental data the following procedure is usually applied. Using an appropriate normalization of the observable quantities, the set of p_i values is determined. Then the generalized dimensions are obtained from

$$D_q = \lim_{\epsilon \to 0} \left[\frac{1}{q-1} \frac{\ln \sum_i p_i^q}{\ln \epsilon} \right] \tag{3.16}$$

with the help of the corresponding log-log plot. Finally, the resulting plot of D_q versus q is numerically derived to obtain $f(\alpha)$.

Another alternative to present experimental results is based on the expressions (3.2) and (3.3) defining α and $f(\alpha)$ through the set p_i. For a fractal measure the plots $\ln[N(p_i, \epsilon)]/\ln(1/\epsilon)$ versus $\ln p_i/\ln(1/\epsilon)$ for various

$\epsilon \ll 1$ should fall onto the same universal curve which is the $f(\alpha)$ spectrum of the multifractal (Meakin 1987). In this approach corrections proportional to $1/\ln\epsilon$ are expected to make the evaluation of data less effective.

Finally, we make a few remarks concerning the order $q = 1$ generalized dimension, for which we have from equation (3.11) and its derivative taken at $q = 1$ the following relation

$$D_1 = \alpha_1 = f_1. \tag{3.17}$$

The above expression will be used in Example 3.1 to explain some of the properties possessed by the plots of $f(\alpha)$, f_q and α_q. Furthermore, inserting (3.5) into (3.7) and taking the $q \to 1$ limit (using l'Hospital's rule) one gets

$$-\sum_i p_i \ln p_i \sim D_1 \ln(1/\epsilon). \tag{3.18}$$

The left hand side of (3.18) is a familiar expression from information theory and corresponds to the amount of information associated with the distribution of p_i values. Therefore, according to (3.18) D_1 describes the scaling of information as $\epsilon \to 0$. This is why D_1 is called *information dimension.* A distribution with $D_1 < D$ is necessarily a fractal measure (Farmer 1982).

3.3. FRACTAL MEASURES CONSTRUCTED BY RECURSION

The formalism described in the previous section assumes the knowledge of p_i, the set of box probabilities. For a real system or computer models p_i has to be determined experimentally or numerically, but in the case of *deterministic fractal measures* constructed by an exact recursive procedure the box probability distribution can be obtained analytically, just like the fractal dimension for ordinary deterministic fractals (Sec. 2.3.1.).

To construct a multifractal distribution we generalize the procedure used in Chapter 2. to produce non-uniform fractals, to the case when the weight or measure corresponding to a newly generated part is also changed by a given factor. Thus, we consider the following construction: in the first step ($k = 1$) the ith part obtained from the seed object by reduction using a rescaling factor $(1/r_j) < 1$ will be given a probability P_j, with $\sum_j^n P_j = 1$, where n is the number of pieces the starting configuration is substituted with. At the next stage ($k = 2$) each part is divided into n parts with probabilities reduced further by P_j and size rescaled $1/r_j$. This procedure is illustrated for $n = 3$ in Fig. 3.1. The multifractal is obtained in the $k \to \infty$ limit.

As a result of the above construction the fractal structure (support) on which the measure is defined can be divided into n parts each being a rescaled version of the whole support by a factor $1/r_j$. The total measure associated with the jth such part is P_j. Therefore,

$$\chi_{q,j}(\epsilon) = \sum_i p_{j,i}^q = P_j^q \chi_q(\epsilon r_j), \tag{3.19}$$

where $\chi_{q,j}(\epsilon)$ is the quantity defined by (3.5) evaluated for the jth part and $p_{j,i}$ is the probability of the ith box in the jth part. For the whole system

$$\chi_q(\epsilon) = \sum_{j=1}^{n} \chi_{q,j}(\epsilon). \tag{3.20}$$

Using (3.7) and (3.19) in (3.20) we get (Hentschel and Procaccia 1983, Halsey *et al* 1986)

$$\sum_{j=1}^{n} P_j^q r_j^{(q-1)D_q} = 1 \tag{3.21}$$

which is an implicit equation for the generalized dimensions D_q.

Depending on the particular choice for q or the P_j and r_j values in (3.21) various special cases can be recovered. As expected, for $q = 0$ (3.21) provides the fractal dimension of the support, since in this limit it becomes identical with (2.13). On the other hand, for general q, but identical rescaling factors $r_j \equiv r$ equation (3.21) can be solved for D_q

$$D_q = \frac{1}{q-1} \frac{\ln\left(\sum_{j=1}^{n} P_j^q\right)}{\ln(1/r)}. \tag{3.22}$$

Multifractal properties are trivially lost, if all P_j-s and r_j-s are equal. Furthermore, distributing the measure on the support with a constant density, i.e. choosing

$$P_j = \frac{r_j^{-1}}{\sum_{j=1}^{n}(1/r_j)} \tag{3.23}$$

results in a non-trivial D_q spectrum which will be discussed in the next Section.

EXAMPLES

Example 3.1. First we consider a multifractal distribution which, in spite of its simplicity, possesses the relevant features of more general fractal measures (Farmer 1982). The $f(\alpha)$ spectrum of this construction can be treated by exact calculations (Tél 1988), in addition to the self-similarity considerations leading to (3.21). The measure will be defined on the unit interval instead of a fractal support, but this fact does not change its characteristic scaling properties.

Thus, consider the unit interval divided into three equal parts of length (1/3), with the corresponding weights or probabilities P_1, P_1 and P_2, where $P_2 = 1 - 2P_1$. We shall assume that the probability of the middle interval is larger than that of the two others having equal weights, i.e., $P_2 > P_1$. In the next step ($k = 2$) each of the three intervals is further divided and the probability is redistributed within the 9 new intervals according to the proportions used in the first step. This construction, which corresponds to the special case $n = 3$ and $1/r_j \equiv 1/3$, is illustrated in Fig. 3.2. In a few more steps the distribution becomes so inhomogeneous that its structure

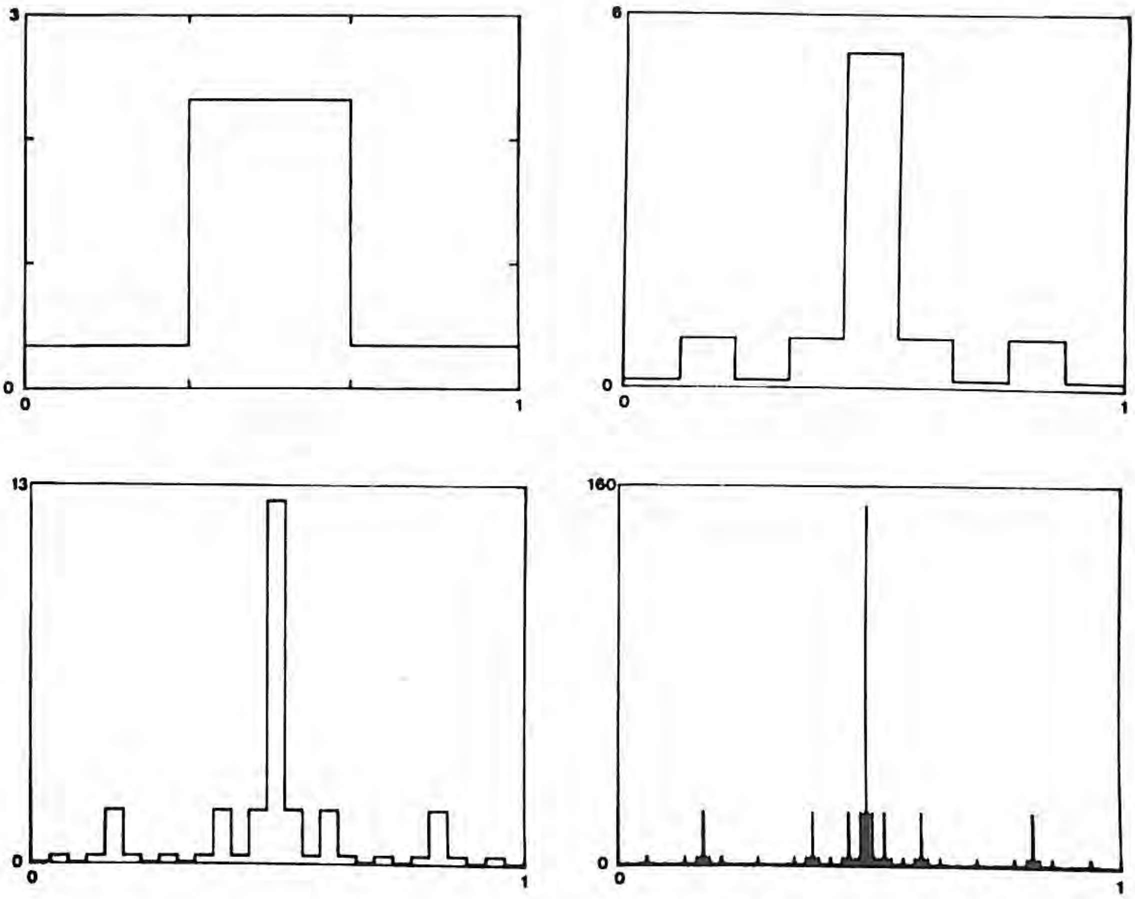

Figure 3.2. The first steps of constructing a fractal measure on the unit interval (Farmer 1982).

becomes visible only on a logarithmic scale. The density distribution in the limit $k \to \infty$ turns into a single-valued, everywhere discontinuous function.

Application of (3.22) yields an explicit expression for D_q through the probabilities P_1 and P_2

$$D_q = \frac{1}{(q-1)\ln(1/3)} \ln(2P_1^q + P_2^q). \tag{3.24}$$

From the above equation α_q can be obtained using (3.15)

$$\alpha_q = \frac{1}{\ln(1/3)} \frac{\ln(2P_1^q \ln P_1 + P_2^q \ln P_2)}{2P_1^q + P_2^q}. \tag{3.25}$$

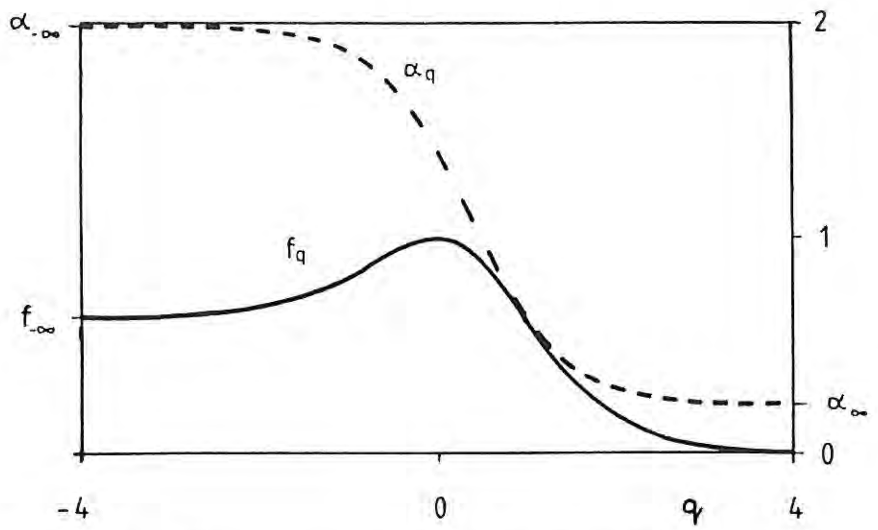

Figure 3.3. The q-dependence of the fractal dimension f_q and the exponent α_q for the multifractal shown in Fig. 3.2 (Tél 1988).

Similarly, we can determine f_q substituting (3.24) and (3.25) into (3.11)

$$f_q = \frac{1}{\ln(1/3)}\left[\frac{2P_1^q \ln P_1^q + P_2^q \ln P_2^q}{2P_1^q + P_2^q} - \ln(2P_1^q + P_2^q)\right]. \tag{3.26}$$

The dependence of f_q and α_q on q is shown in Fig. 3.3. Knowing f_q and α_q we can determine $f(\alpha)$ from (3.14). The result is displayed in Fig. 3.4.

To demonstrate how the singularity strength α_q with the corresponding fractal dimension f_q is selected for a given q we determine the distribution of p_i-s explicitly (without using (3.22)). Let us cover the unit interval with one dimensional boxes of size $l = \epsilon = (1/3)k$ and assume that $k \gg 1$. According to the construction of the present example the possible values of the box probabilities are the following

$$p_m = P_1^m P_2^{k-m}, \tag{3.27}$$

where m is an integer number and $0 \leq m \leq k$. The number of boxes with probability p_m is given by

$$N_m = \binom{k}{m} 2^m. \tag{3.28}$$

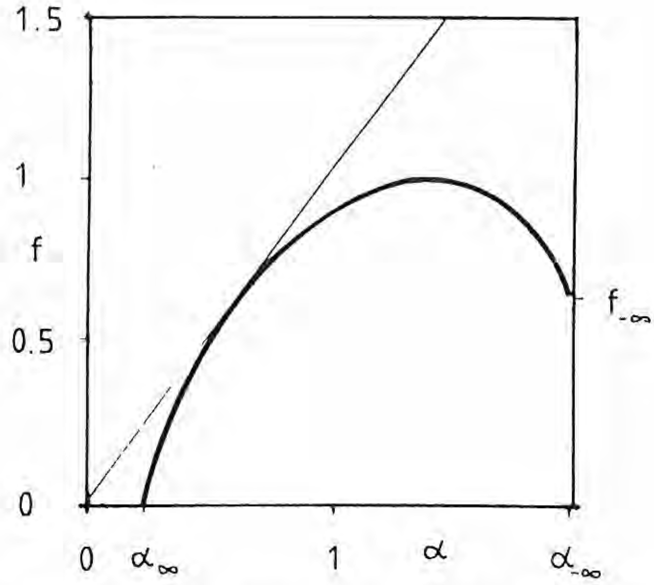

Figure 3.4. The $f(\alpha)$ spectrum of fractal dimensionalities for the multifractal of Fig. 3.2. The straight line corresponds to $f = \alpha$ (Tél 1988).

Next we investigate which set of the boxes gives a finite contribution to the total probability in the limit $k \to \infty$. For this purpose we calculate $\ln N_m p_m$ using Stirling's formula $\ln n! \simeq n(\ln n - 1)$ and get

$$\ln N_m p_m = k \ln k - m \ln m - (k-m) \ln(k-m) + m \ln 2 + m \ln P_1 + (k-m) \ln P_2. \tag{3.29}$$

The maximum of (3.29) is at

$$m_1 = 2kP_1, \tag{3.30}$$

and the value of $\ln N_{m_1} p_{m_1}$ is equal to 0 within the accuracy of Stirling's formula. This means that the contribution of boxes with m_1 is approximately equal to 1, therefore these boxes contain nearly all of the total measure. As concerning boxes with other values of m, their contribution is negligible, because either their number or the measure they carry is too small.

To find the fractal dimension of the structure made of boxes with p_{m_1} we first express $\ln N_{m_1}$ using (3.28) and (3.30)

$$\ln N_{m_1} = -k(2P_1 \ln P_1 + P_2 \ln P_2). \tag{3.31}$$

Then, from (3.31) and $\epsilon = (1/3)^k$ it follows that

$$N_{m_1}(\epsilon) = \epsilon^{-f_1} \tag{3.32}$$

with

$$f_1 = \frac{1}{\ln(1/3)}(2P_1 \ln P_1 + \ln P_2). \tag{3.33}$$

Therefore, boxes forming a fractal of dimension f_1 give the dominant contribution to the stationary distribution.

The following step is to study the qth power of the box probabilities with $-\infty < q < \infty$. Their total amount is $\sum_m N_m p_m^q = (2P_1^q + P_2^q)^k$. As before, the dominant contribution is given by boxes with a characteristic probability, p_{m_q}, depending on q. Simple algebra analogous to the above derivation leads to (Tél 1988)

$$m_q = \frac{2kP_1^q}{2P_1^q + P_2^q}$$

Writing N_{m_q} and P_{m_q} in the form $N_{m_q}(\epsilon) = \epsilon^{-f_q}$ and $P_{m_q}(\epsilon) = \epsilon^{\alpha_q}$ it is easy to show that these expressions hold with f_q and α_q identical with those obtained by the similarity argument (see Eqs. (3.25) and (3.26)).

Example 3.2. In order to discuss an example which has direct connection with physical systems let us consider a hierarchical network of resistors shown in Fig. 3.5. At the kth step of the iteration procedure each bond of length $l = 1$ is replaced with a configuration having a linear size equal to $r = 3$ and made of $n = 6$ bonds of unit length. This growing fractal in the $k \to \infty$ limit has a fractal dimension $D = \ln 6/\ln 3$. To complete the model, we assume that the bonds have the same resistivity and the voltage between the two terminal bonds is equal to $V_0 = 1$. Then, in the unit cell there will be two bonds with voltage drop $V_1 = 1/3$ and four others with $V_2 = 1/6$. The sum of all voltage drops is not equal to 1, therefore, the distribution is not normalized. (It can be normalized by simply choosing

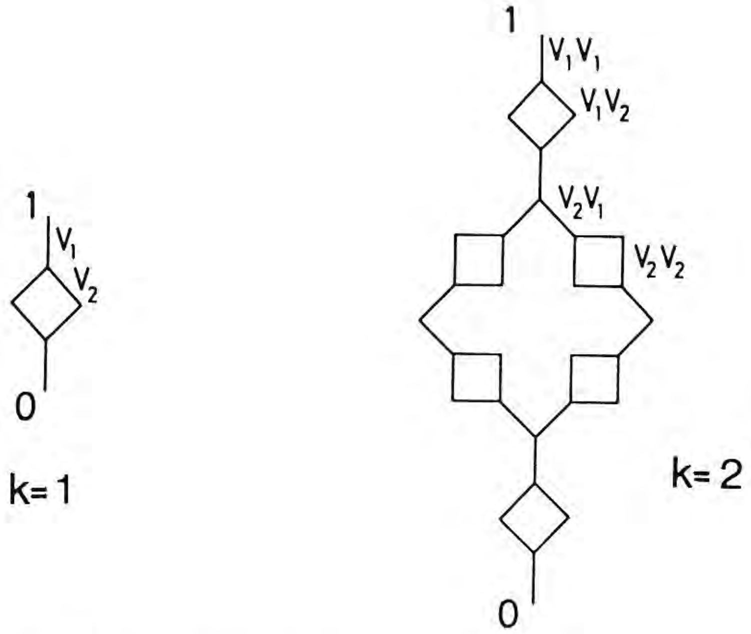

Figure 3.5. Fractal model for hierarchical networks of resistors. The distribution of voltage drops exhibits multifractal scaling.

$V_0 = 3/4$.)

The multifractal properties can be deduced from (Arcangelis *et al* 1985, 1986)

$$\sum_i v_i^q \sim L^{-(q-1)D_q}, \qquad (3.34)$$

where v_i is the voltage drop across the ith bond and $L = (l/\epsilon) = 3^k$ is the linear size of the network. Because of the hierarchical structure of the model the sum of the v_i^q-s after completing the kth recursion is

$$\sum_m N_m v_i^q = (2V_1^q + 4V_2^q)^k. \qquad (3.35)$$

In the above expression

$$N_m = 2^{k+m}\binom{k}{m} \qquad (3.36)$$

is the number of bonds with the voltage drop

$$v_m = V_1^{k-m} V_2^m \tag{3.37}$$

with $0 \leq m \leq k$. Substituting (3.35) into (3.34) and using the actual voltage values we get

$$D_q = \frac{1}{q-1} \frac{\ln[2\left(\frac{1}{3}\right)^q + 4\left(\frac{1}{6}\right)^q]}{\ln(1/3)} \tag{3.38}$$

showing that the moments of the distribution of voltage drops can be described by an infinite number of generalized dimensions. Finally, we remark that the distribution (3.36) as a function of m is a simple binomial modified by a factor 2^m. Since the voltage drops depend on m approximately exponentially, we conclude that the distribution $N(v_m)$ has a rather special shape which is close to the log-binomial. Finally, it should be noted that using an appropriately modified definition of the exponents corresponding to the moments of the voltage distribution it is possible to obtain an infinite hierarchy of exponents which is independent of the actual fractal dimension of the network (i.e., which reflects only its topological properties).

3.4. GEOMETRICAL MULTIFRACTALITY

The results discussed in the previous two Sections were obtained for the general case when an inhomogeneous measure was defined on the support. The special case of a distribution with constant density on a non-uniform fractal, however, deserves particular attention. In this case it is the geometry of the fractal only which is described by the formalism. Since this book concentrates on the structure of growing fractals rather than on their physical properties, in the following the application of the theoretical approach developed above will be discussed for structures with no singular density defined on them.

The fact that non-uniform fractals with a uniform distribution can also be described in terms of generalized dimensions has already been indicated by expression (3.21). This equation valid for a recursive fractal leads to a

D_q spectrum even if the weight factors P_j are defined in such a way that the measure of a newly created part becomes the same as its volume, i.e., the probability distribution is uniform on the support. In the case of Example 3.1. this can be achieved by choosing $P_j = (1/r_j)/\sum_j(1/r_j)$ which together with (3.21) results in the non-trivial dependence of D_q on q.

Let us now consider the problem for the general case of growing fractals. We assume that the structures grow on a lattice and are built up by identical particles. The actual linear size and mass of the cluster will be denoted by L and M, respectively. The structure is to be covered by boxes of size l such that

$$a \ll l \ll L, \tag{3.39}$$

where a is the lattice constant. (In some cases the condition $a/l \to 0$ has to be satisfied.) One can then determine the mass M_i of the i-th non-empty box. The mass index or singularity exponent α of this box is defined by

$$M_i \sim M(l/L)^\alpha \tag{3.40}$$

for $l/L \ll 1$. Boxes with the same mass index α form a fractal subset of dimension $f(\alpha)$. Their number $N(\alpha)$ is, therefore, related to $f(\alpha)$ via

$$N(\alpha) \sim (l/L)^{-f(\alpha)}. \tag{3.41}$$

If there exists a set of different mass indices the growing structure will be called a *geometrical (or mass) multifractal* (Tél and Vicsek 1987), since the measure generating the spectrum is a uniform mass distribution (Lebesque measure), and thus $f(\alpha)$ characterizes the geometry of the system directly.

To determine the generalized dimensions one can use the scaling relation (3.7) which in the present notation has the form

$$\chi(M,l,L) \equiv \sum_i M_i^q \sim M^q(l/L)^{(q-1)D_q}. \tag{3.42}$$

As for general multifractals, knowing D_q we can determine $f(\alpha)$ using expressions (3.11) and (3.13).

Before discussing the examples one should make a few important comments. i) Non-uniform recursive fractals are multifractals in a geometrical sense. ii) Since geometrical multifractality as defined above is a consequence of local density fluctuations, it should be most pronounced in inhomogeneous growth processes. iii) A necessary condition for observing this phenomenon is the existence of three well separated length scales (see (3.39)) which may require the linear size L to be close to the largest cluster size ever produced in numerical simulations.

Finally, one can simply check that in the case of the examples to be discussed below the values obtained for the fractal dimension using two different methods, *do not coincide*. First, one can apply the so called sandbox method which is equivalent to (2.4), i.e., to calculating the number of particles $M(L)$ within boxes of increasing linear size L centred at the same point. Applying this method at each step of the construction of a non-uniform fractal (2.9) gives a fractal dimension different from that obtained from equation (2.13) derived for the fractal dimension from self-similarity arguments. In fact, the dimension obtained from the sandbox method is equal to $D_{-\infty}$. The contradiction is resolved by the unusual behaviour of $\ln M(L)$ versus $\ln L$ as a function of the position chosen for the centre of boxes. Because of the extreme inhomogeneity of the structure the quantity

$$\ln M(L)/\ln L \tag{3.43}$$

which for large L usually is equal to the fractal dimension does not converge to a unique limiting value as $L \to \infty$. The right fractal dimension can be obtained by averaging over the position of sandbox centres.

EXAMPLES

Example 3.3 The construction we shall investigate is a growing version of the non-uniform Cantor set (Fig. 3.6). This growth process embedded into one dimension enables us to determine how the multifractal behaviour emerges as the system grows (Tél and Vicsek 1987). The first unit of the process consists of three particles in the first, third and fourth sites. At the next stage the twice enlarged copy of the "seed" configuration is added between the 9th and 16th sites (leaving out four sites equal to the length of the first configuration) and this procedure is repeated with the new configuration playing the role of the seed. Fig. 3.6 shows the objects obtained in the first four steps of the construction.

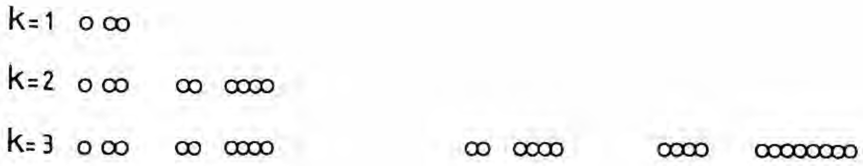

Figure 3.6. The first few steps in the construction of the growing non-uniform Cantor set (Tél and Vicsek 1987).

After k steps the linear size is 4^k, it is, therefore, convenient to use $l = 2^k$ as the box size. It can be easily observed that the number of nonempty boxes is then F_k, a Fibonacci number defined by $F_o = 1$, $F_1 = 2$, $F_k = F_{k-1} + F_{k-2}$. Let $M_i^{(k)}$, $i = 1, ...F_k$ denote the mass of the i-th nonempty box at the kth step of construction (i=1 corresponds to the leftmost box). The distribution $M_i^{(k)}$ is related to that of two previous generations for $k > 2$ by

$$M_i^{(k)} = \begin{cases} 3M_i^{(k-2)} & i = 1, ...F_{k-2}, \\ 2M_{F_{k-2}+i}^{(k-1)} & i = 1, ...F_{k-1}, \end{cases} \tag{3.44}$$

($M = 3^k$) as can be checked directly. The "initial values" for this recursion are $M_i^{(1)} = 1, 2$, $M_i^{(2)} = 3, 2, 4$. At this point it is already possible to calculate a few characteristic exponents for the system. Since the number of nonempty boxes is F_k and $F_{k+1}/F_k \approx w$ for large k, where $w = (\sqrt{5}+1)/2$ is the golden mean, we find from (3.42) $D_o = ln(w)/ln(2)$ for the fractal dimension of the complete set. The densest box is the rightmost one with mass 2^k, therefore, the smallest α is obtained from (3.40) as $\alpha_{min} = D_{\infty} = ln(2/3)/ln(1/2) = 0.585...$. The most rarified nonempty interval is the first or the second one. For its mass we find $3^{(k-1)/2}$ for an odd k and $2 \times 3^{k/2-1}$ for even k. Consequently, the largest α is $\alpha_{max} = D_{-\infty} = ln(1/3)/ln(1/4) = 0.792....$

The picture simplifies further by observing that there are only $k+1$ different mass values in the set $M_i^{(k)}$. Denoting these quantities by $\tilde{M}_j^{(k)}$, $j = 1, ...k+1$ one obtains for $k > 1$

$$\tilde{M}_j^{(k)} = \begin{cases} 3^{[k/2]} & j = 1 \\ 2M_{j-1}^{(k-1)} & j = 2, ..., k+1 \end{cases} \tag{3.45}$$

with $\tilde{M}_j^{(1)} = 1, 2$ where [ ] denotes the integer part. Let the numbers $N_j^{(k)}$ denote how many times the value $\tilde{M}_j^{(k)}$ appears in $M_j^{(k)}$. They are found to follow a two step recursion

$$N_j^{(k)} = N_{j-1}^{(k-1)} + N_j^{(k-2)}, \qquad 1 < j < k \tag{3.46}$$

and $N_j^{(k)} = N_k^{(k)} = N_{k+1}^{(k)} = 1$. Now it is possible to derive an equation for the $f(\alpha)$ spectrum using (3.40), (3.41), (3.45) and (3.46). The solution is

$$f(\alpha) = \left[(\alpha - \alpha_{min}) ln\left(\frac{\alpha - \alpha_{min}}{1-\alpha}\right) + 2(\alpha_{max} - \alpha) ln\left(\frac{2(\alpha_{max} - \alpha)}{1-\alpha}\right)\right] \times \frac{1}{2\ln(1/2)(\alpha_{max} - \alpha_{min})}. \tag{3.47}$$

Finally, one can obtain from (3.11) and (3.47) an expression for the D_q spectrum.

The knowledge of recursions (3.45) and (3.46) makes it possible to calculate χ_q numerically for any finite stage, k, of the construction. Comparing two subsequent stages of the growth process we define a k *dependent* $D_q(k)$ by

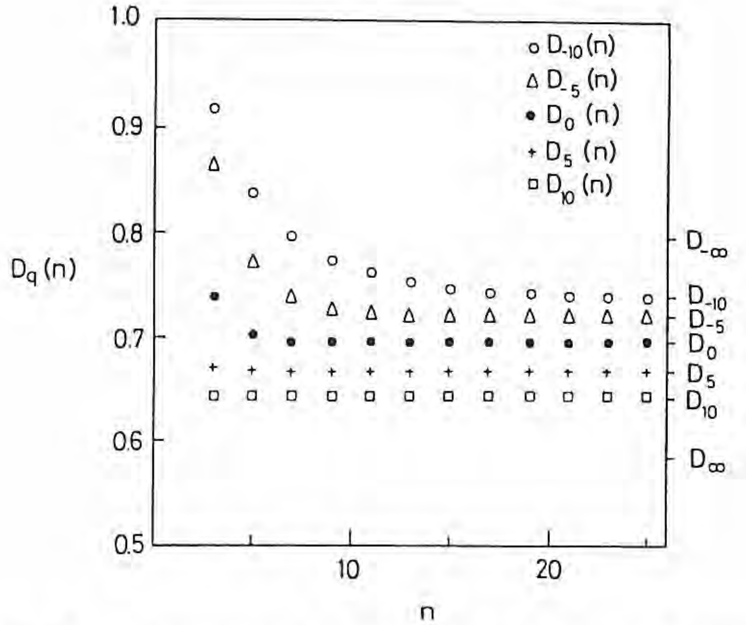

Figure 3.7. Convergence towards the generalized dimensions as a function of the actual size of the growing multifractal (Tél and Vicsek 1987).

$$\frac{\chi_{q,k+1}/M_{k+1}^q}{\chi_{q,k}/M_k^q} = \left(\frac{l_{k+1}/L_{k+1}}{l_k/L_k}\right)^{(q-1)D_q(k)}, \tag{3.48}$$

where the subscript denotes quantities at the k-th stage of the growth. For large k $D_q(k) \to D_q$ as can be seen from (3.42). The results plotted in Fig. 3.7. show a rapid convergence as a function of n towards D_q, but note that the linear size in this model grows as 4^k.

It is also possible to calculate numerically the function

$$f_j^{(k)} = -ln(N_j^{(k)})/ln(l_k/L_k) \tag{3.49}$$

versus $\alpha_j^{(k)} = ln(\tilde{M}_j^{(k)}/M_k)/ln(l_k/L_k)$ for *increasing values of k*. These quantities converge to $f(\alpha)$ and α as was shown above. The convergence is, however, slow since the constant factors not written out explicitly in Eqs. (3.40) and (3.41) lead to contributions proportional to k^{-1}. The result for $k = 100$ together with the exact solution is displayed in Fig. 3.8.

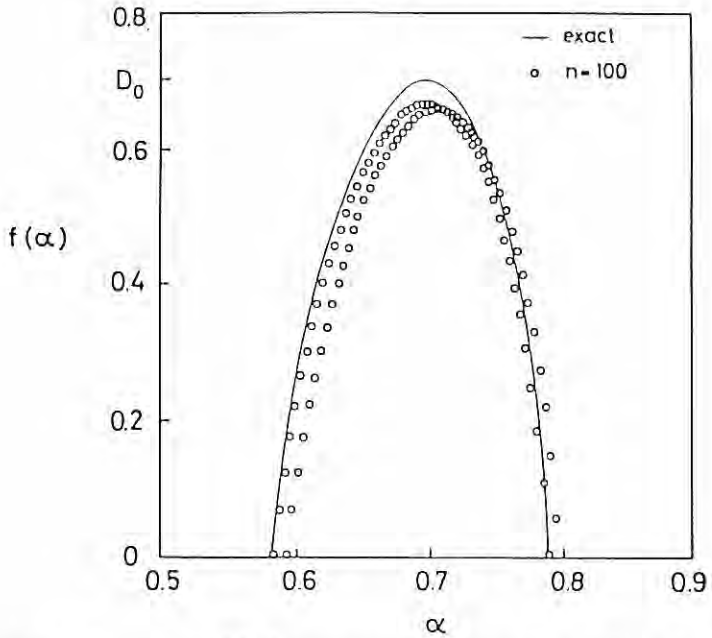

Figure 3.8. The $f(\alpha)$ spectrum obtained from (3.47) is shown by the full curve. The open circles represent the function $f_j^{(k)}$ versus $\alpha_j^{(k)}$ for $k = 100$. The special doubled internal structure of the latter plot is a consequence of the regular geometry of the model (it would be destroyed for a random configuration) (Tél and Vicsek 1987).

Example 3.4 To demonstrate geometrical multifractality for a growing fractal embedded into two dimensions we generalize the construction shown in Fig. 2.1a. The recursion procedure leading to a non-uniform fractal cluster is explained in Fig. 3.9. In the kth stage the twice enlarged version of the configuration corresponding to the $k-1$th stage of growth is added to the four corners of the cluster obtained for $k-1$. Although the large homogeneous plaquettes at stage k have an increasing size 2^{k-1} their size relative to that of the cluster is $(2/5)^{k-1}$, approaches zero as $k \to \infty$.

We shall use (3.21) to get an implicit equation for the generalized dimensions. First note that at each step the configuration is replaced by five new parts, one of which has the same size as the previous configuration and 4 are twice enlarged versions of it. Simultaneously, the linear size of the cluster becomes 5 times larger. Let us reduce the growing structure at every step by a factor 1/5 so that (3.21) could be applied directly with rescaling factors to $1/r_1 = 1/5$ and $1/r_2 = ... = 1/r_5 = 2/5$. Furthermore, the part

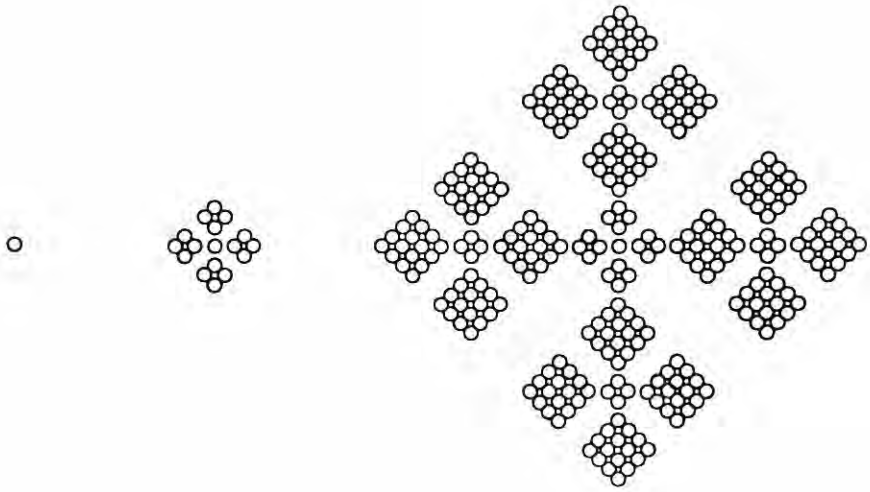

Figure 3.9. Growing geometrical multifractal embedded into two dimensions. In the kth step the twice enlarged copy of the previous stage is added to the four corners of the configuration.

with r_1 has a mass which is $P_1 = 1/17$ times smaller than that of the whole structure. Analogously, $P_2 = ... = P_5 = 4/17$. Therefore, we have

$$\sum_{j=1}^{5} r_j^{(q-1)D_q} P_j^q = \left(\frac{1}{17}\right)^q 5^{(q-1)D_q} + 4\left(\frac{4}{17}\right)^q \left(\frac{5}{2}\right)^{(q-1)D_q} = 1. \quad (3.50)$$

The above equation can be solved for general q numerically, while in the limits $q \to \pm\infty$ it provides the explicit expressions $D_\infty = \ln \frac{17}{4} / \ln \frac{5}{2} \simeq 1.579$ and $D_{-\infty} = \ln 17 / \ln 5 \simeq 1.760$. For $-\infty < q < \infty$ the corresponding D_q-s are between the above limiting values. Note that the sandbox method (i.e., calculating the number of particles within boxes of size 5^k centred at the seed particle) would incorrectly predict $D = D_{-\infty}$.