Opinion dynamics
&
Synchronization

Bioinspired Systems - Oct 13, 2022

Opinion dynamics (cont.)

- The scientific field aiming to understand the way „opinions” spread in human communities.
  - The community is usually described by means of networks
    - Nodes are individuals
    - Links are the ties (connections)
      - Direction
      - Strength
  - 3 main types of models:
    - Opinions are binary (-1/1)
    - Opinions are continuous [0, 1]
    - Agents have „inner structure” (including multidimensional vector)

Cultural dynamics

Multidimensional vector model

- Mostly: opinion: scalar variable

„culture”: a vector of variables

The typical questions are similar:

- what are the microscopic mechanisms that drive the formation of cultural domains?
- What is the ultimate fate of diversity?
- Is it bound to persist or all differences eventually disappear in the long run?
- What is the role of the social network structure?

Axelrod model

- Axelrod, 1997
- Attracted lot of interest both from social scientists and physicists
  - Reason (soc. sci): inclusion of two fundamental mechanisms:
    - Social influence: the tendency of individuals to become more similar when they interact
    - Homophily: the tendency of alikes to attract each other, so that they interact more frequently
  - These two ingredients were generally expected to generate a self-reinforcing dynamics leading to a global convergence to a single culture.
  - But it turns out that the model predicts in some cases the persistence of diversity. (The importance of minimal models!)
  - From the viewpoint of stat. phys:
    - is a "vectorial" generalization of opinion dynamics models
    - gives rise to a very rich and nontrivial phenomenology, with some genuinely novel behavior

Axelrod’s model

- **Individuals** :
  - are nodes on a network (or on the sites of a regular lattice – original version)
  - They are endowed with F integer variables ($\sigma_1, \ldots, \sigma_F$) (describing their “culture”)

    The variables are the “*cultural features*”
- **Each σ_i** (feature) can assume q values: $\sigma_f = 0,1,\ldots,q-1$
  - q: number of possible traits (modeling the different “beliefs, attitudes and behavior” of individuals
- **An elementary step**:
  - an individual i and one of his neighbors j are selected
  - The overlap between them is computed:

$$\omega_{i,j} = \frac{1}{F} \sum_{f=1}^{F} \delta_{\sigma_f(i),\sigma_f(j)}$$

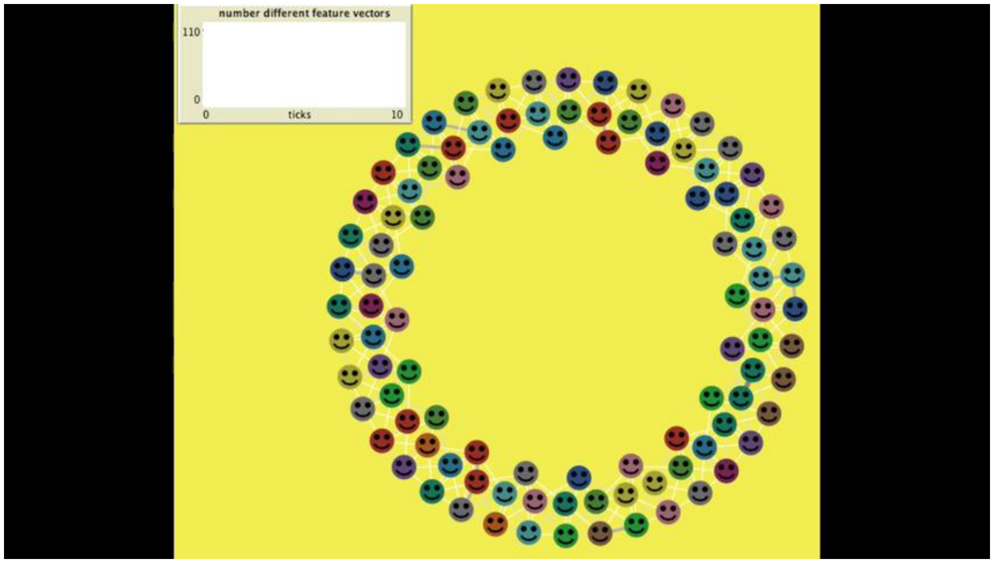
    Where $\delta_{i,j}$ is the Kronecker delta

  - $\omega_{i,j}$: probability of interaction between i and j
    - If they interact: one of the features for which traits are different ($\sigma_f(i) \neq \sigma_f(j)$) is selected and the trait of the neighbor is set equal to σ_f (i)
    - If they do not interact: nothing happens

Features of the Axelrod model

- the dynamics tends to make interacting individuals more similar
- Interaction:
  - more likely for neighbors already sharing many traits (homophily)
  - becomes impossible when no trait is the same
- For each pair of neighbors: two stable configurations:
  1. when they are exactly equal, so that they belong to the same cultural region or
  2. when they are completely different, i.e., they sit at the border between cultural regions
- Starting from a disordered initial condition:
  - The evolution on any finite system leads to one of the many **absorbing states**, which belong to **two classes**:
  1. the **ordered states**, in which **all** individuals have the same set of variables, or
  2. **Frozen states** with different coexisting cultural regions (more numerous)
- Which one is reached: depends on *q* (number of possible traits):
  - Small *q*: quickly full consensus is achieved
  - Large q: very few individuals share traits → few interactions occur → formation of small cultural domains that are not able to grow (disordered frozen state)

Axelrod's model of cultural dissemination in a circle network (16 sec)



Inset: number of different feature vectors

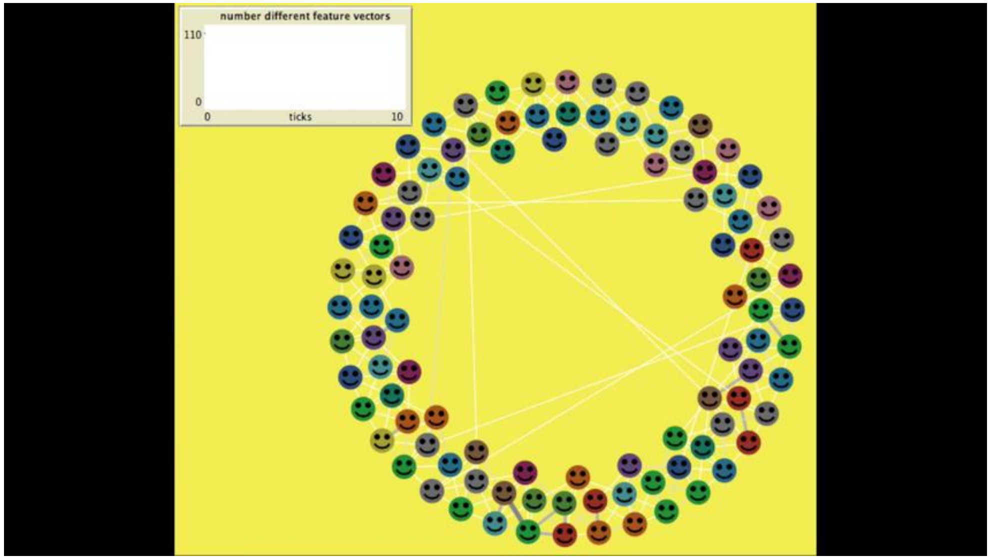
- a circle interaction structure
- 100 agents, each
  - with 6 network contacts
  - 5 features
- Each feature can adopt 15 values
- Init: random set of traits
- Color of agents: *one* of the features
- Color and thickness of the lines: the overall similarity between the respective nodes
  - black thick lines: identical traits on all features
  - White thin lines: the two nodes are connected but maximally different.

- emergence of internally homogenous but mutually different clusters.
- Dynamics settled after 34,809 iterations with 19 cultural clusters.

(Michael Maes, 2015)

https://www.youtube.com/watch?v=kLPIeQAKQQw

Axelrod's model of cultural dissemination in a small world network (47 sec)



- a small-world interaction structure
- 100 agents, each
  - with 6 network contacts
  - 5 features
- Each feature can adopt 15 values
- Init: random set of traits
- Color of agents: *one* of the features
- Color and thickness of the lines: the overall similarity between the respective nodes

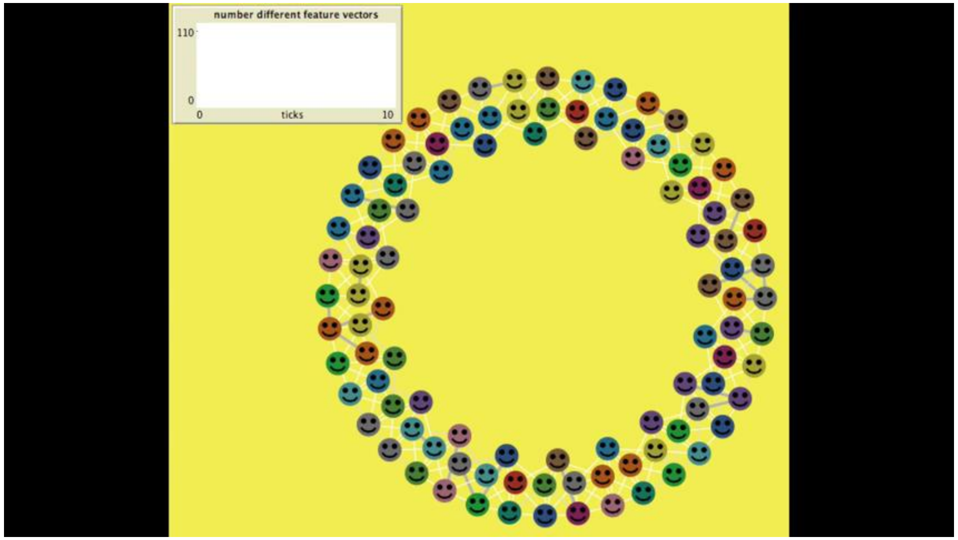
- emergence of internally homogenous but mutually maximally different clusters.
- Dynamics settled after 140,427 iterations with 7 cultural clusters.

Inset: number of different feature vectors

(Michael Maes, 2015)

https://www.youtube.com/watch?v=7ZbVUNWrLYs

Playing with Axelrod's model: the effect of globalization (53 sec)



- Globalization: more individuals are in contact with others who are geographically very distant

- Circle NW interaction structure (at the beginning!)
- 100 agents, each
  - with 6 network contacts
  - 5 features
- Each feature can adopt 15 values
- Init: random set of traits
- Color of agents: *one* of the features
- Color and thickness of the lines: the overall similarity between the respective nodes

- The dynamics reaches a rest point (after 51,065 iterations)
- Rewire 20 links and cont. (modeling that individuals have more contact to distant others)

(Michael Maes, 2015)

https://www.youtube.com/watch?v=VvXjk8P4TX0

Illustrates two implications of the model:

1. due to the rewiring the number of clusters in equilibrium decreased from 22 to 16
2. after the simulation continued (after rewiring) the number of unique combinations of cultural traits (diversity) first increased and then decreased

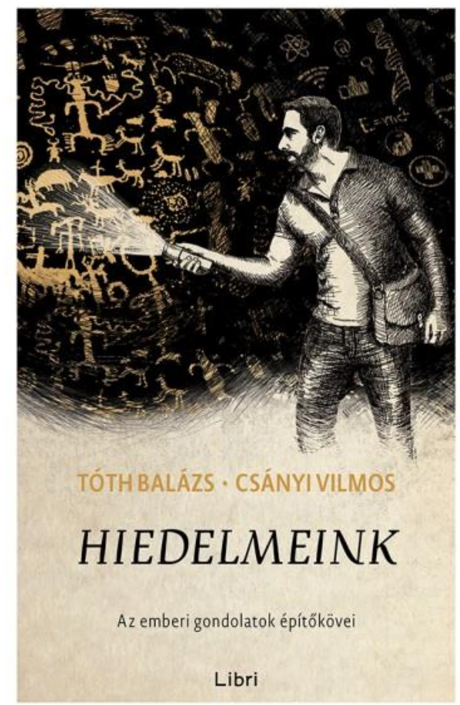
(i) globalization decreases cultural diversity

(ii) the short-term effects differ from the long-term effects

When the „elements of the belief system" (that is: „beliefs") are interrelated

Most models assuming interrelated beliefs are coming from the political science

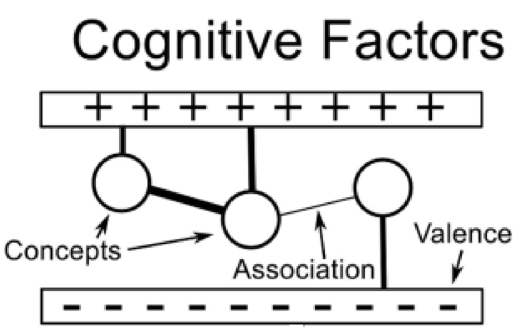
how people form their political attitudes



Two crucial aspects of belief dynamics

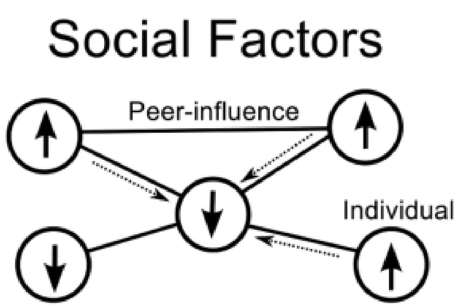
Cognitive bias (or belief bias):

- **Def**: A person's tendency to accept arguments that supports a conclusion that aligns with his/her values, beliefs and prior knowledge, while rejecting counter arguments to the conclusion
- Leads to individual belief rigidity
- Cognitive dissonance (well-studied area)



Social influence:

- The tendency of individuals to become more similar when they interact (we have seen it at the Axelrod model)
- Leads to social conformity

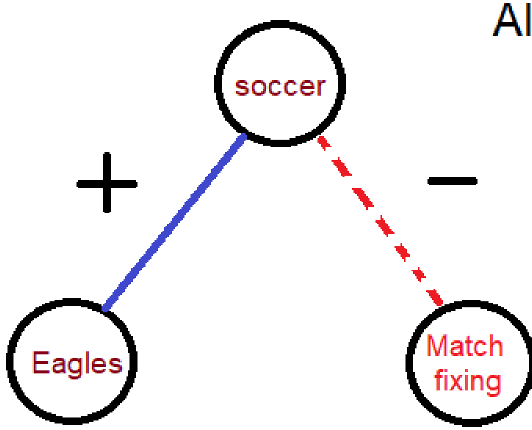


Rodriguez N, Bollen J, Ahn Y-Y (2016) Collective Dynamics of Belief Evolution under Cognitive Coherence and Social Conformity. PLoS ONE 11(11): e0165910. doi:10.1371/journal.pone.0165910

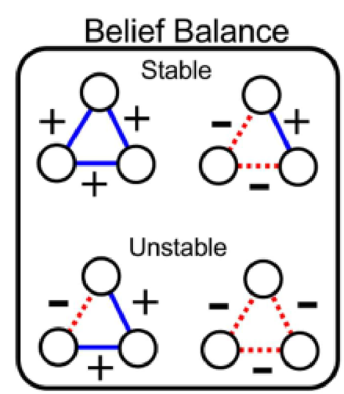
A cognitive-social model

- Individuals are embedded into a social NW, and social influence takes place via the social ties
- Each individual possesses a *network* of concepts and beliefs
- The internal (in)coherence of each individual's belief network is evaluated

The belief system of Alex, the soccer fanatic



1) Alex loves soccer
2) Alex supports the Eagles



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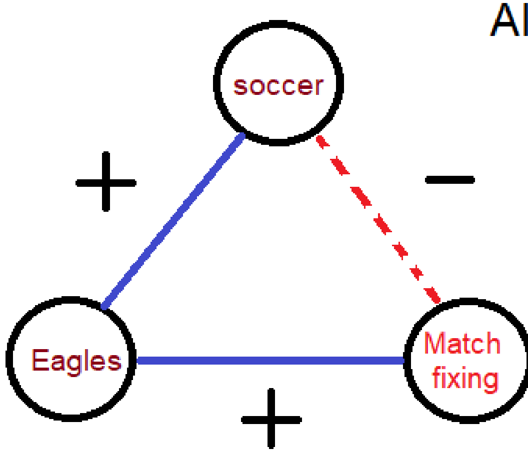
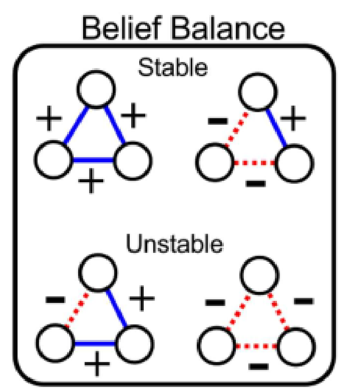


1) Alex loves soccer
2) Alex supports the Eagles
3) The eagles has been with match fixing



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A cognitive-social model

- The *internal coherence* of each individual's belief network is evaluated by the *internal energy function*

(on the belief NW *M*):

(For simplicity, the belief NW is complete, meaning that all concepts have a positive or negative association with every other)

$$E_n^{(i)} = -\frac{1}{\binom{M}{3}} \sum_{j,k,l} a_{jk} a_{kl} a_{jl}$$

- The evolution of belief systems is also driven by social interactions: *social energy* term, capturing the *degree of alignment* between connected individuals.)

$$E_n^{(s)} = -\frac{1}{k_{max} \binom{M}{2}} \sum_{q \in \Gamma(n)} \vec{S}_n \cdot \vec{S}_q$$

k_{max} is a normalization factor, maximum degree of N.

S: belief state vector: each element corresponds to an *edge*

- **Total energy**:

where

I: peer-influence ,

J: coherentism

$$H = \sum_{n \in \mathcal{N}} \left[J E_n^{(i)} + I E_n^{(s)} \right]$$

- The status of the entire society is characterized by

(i) the average internal coherence of the individuals <E$^{(i)}$>, and

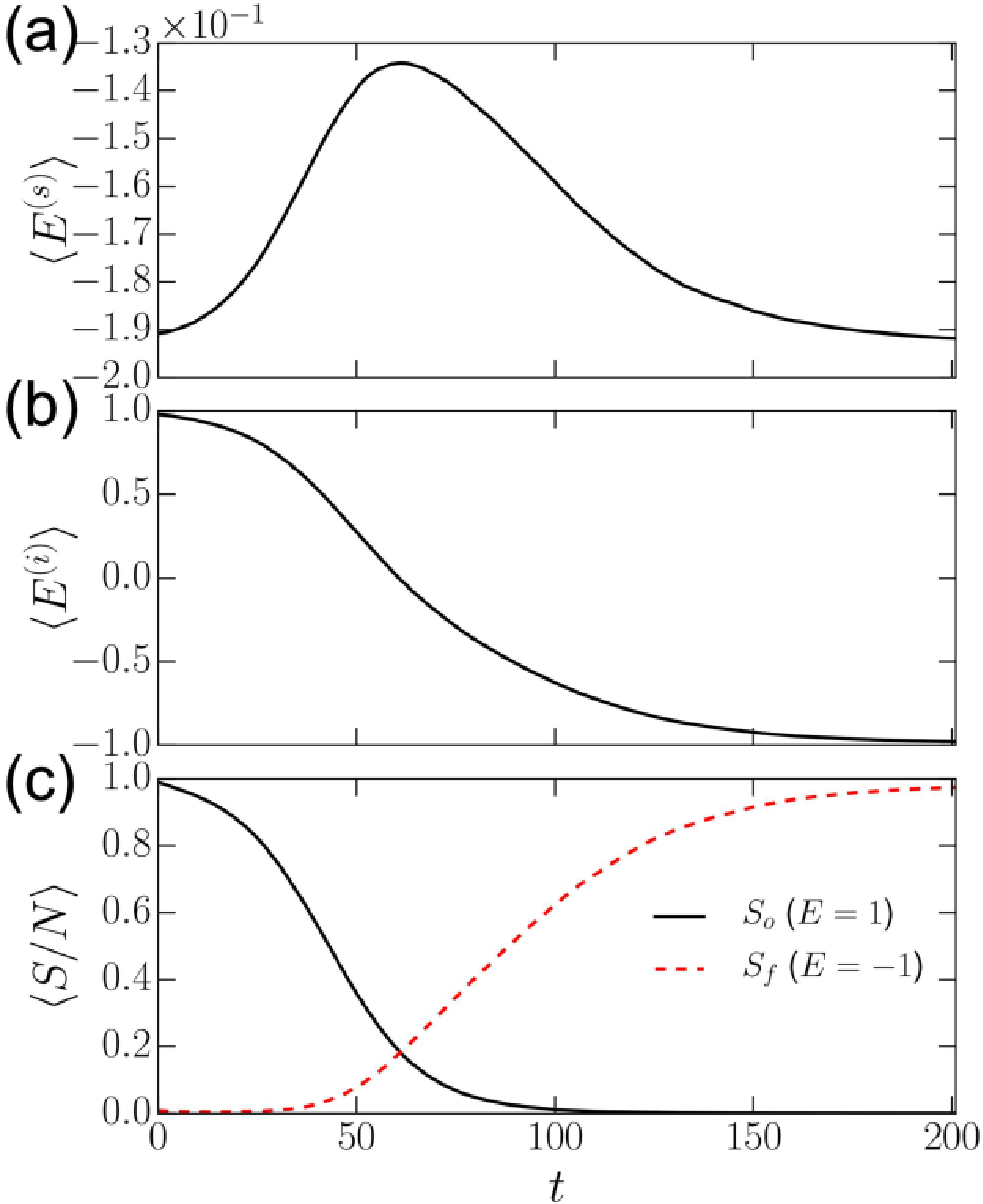
(ii) the homogeneity of the society <E$^{(s)}$>

- The simulation:
  - At each time step a random pair of individuals is chosen
  - One of the individuals (sender)randomly chooses a belief (association) from its internal belief system and sends it to the other individual (receiver)
  - Assumption: each individual has an identical set of concept nodes
  - The receiver accepts it if it decreases its individual energy H_n
  - If $\Delta H_n > 0$, the receiver accepts it with probability $e^{\frac{-\Delta H_n}{T}}$
  - T is “susceptibility” / “open-mindedness”

Results

- Given a *homogeneous* population of people with *highly coherent* belief systems, society remains stable.
- Given a *homogeneous* population of *incoherent belief systems*, society will become unstable and following a small perturbation, breaks down

- In simulation:
  - The society is initialized at **consensus** with an **incoherent** belief system.
  - Then 1% of the population are given a random belief system
  - Individuals attempt to reduce the energy of their own belief systems and leave consensus



In the simulation, the society is initialized at consensus with an incoherent belief system. Then 1% of the population are given a random belief system.

Strong societal consensus does not guarantee a stable society in our model. If major paradigm shifts occur and make individual belief systems incoherent, then society may become unstable.

(a) The plot shows the evolution of social energy $E^{(s)}$ over time. The system starts at consensus but with incoherent beliefs. After introducing a small perturbation, individuals leave consensus, searching for more coherent sets of beliefs, until society reconverges at a stable configuration.

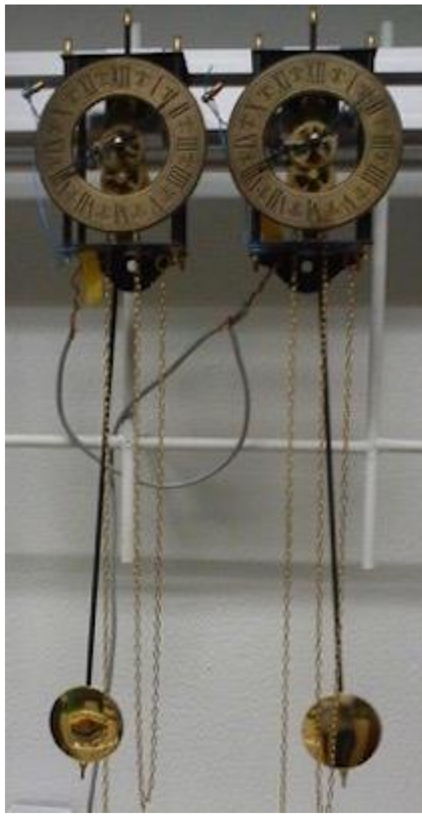
(b) Decreasing mean individual energies $\langle E_{(i)} \rangle$ over time illustrates individual stabilization during societal transition.

(c) $\langle S/N \rangle$ is the fractional group size. As society is upset, the original dominant but incoherent belief system S_o (solid black) is replaced by an emerging coherent alternative S_f (dashed red).

Biological Synchronization

First example of spontaneous synchronization

- Huygens, 1665
- Inventor of pendulum clocks
- Hang two clocks to the same wall
- In half an hour they always regained synchrony
- Opposite wall: one loosing 5 sec a day relative to the other
- *Theory of coupled oscillators*



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Published online 21 February 2002 | Nature | doi:10.1038/news020218-16

News

Synchronized swinging

Ancient pendulum conundrum solved.

John Whitfield

Twin pendulums mirror each other's movements.

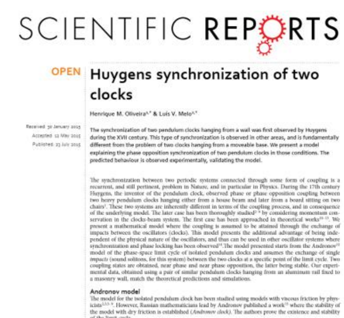
US physicists have solved a 350-year-old riddle of why the pendulums of two clocks become synchronized. The clocks were the first example of spontaneous synchronization, a phenomenon found throughout nature from cells to the Solar System.

In 1665, the Dutch physicist Christiaan Huygens, who patented the pendulum clock, was working on a way to calculate longitude on board ships. His approach was to put two clocks in a single housing, so that if one broke down, the other could provide a back-up.

Ill in bed, with nothing better to do than watch his clocks, Huygens noticed that no matter where they started, within about half an hour the two pendulums swung towards and away from each other in unison. His observations - made 20 years before Newton devised the laws of mechanics - have gone unexplained until now.

Physicist Michael Schatz and his colleagues at the Georgia Institute of Technology reconstructed Huygens' two-in-one clock box from his lab notes and diagrams.

They found that the clocks' behaviour resulted from the 40 kg of lead ballast in the apparatus, intended to hold the clocks fast at sea.



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Huygens synchronization of two clocks

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The synchronization of two pendulum clocks hanging from a wall was first observed by Huygens during the XVII century. This type of synchronization is observed in other areas, and is fundamentally different from the problem of two clocks hanging from a moveable base. We present a model explaining the phase opposition synchronization of two pendulum clocks in those conditions. The predicted behaviour is observed experimentally, validating the model.

The synchronization between two periodic systems connected through some form of coupling is a recurrent, and still pertinent, problem in Nature, and in particular in Physics. During the 17th century Huygens, the inventor of the pendulum clock, observed phase or phase opposition coupling between two heavy pendulum clocks hanging either from a house beam and later from a board sitting on two chairs. These two systems are inherently different in terms of the coupling process, and in consequence of the underlying model. The later case has been thoroughly studied by considering momentum conservation in the clocks-beam system. The first case has been approached in theoretical works. We present a mathematical model where the coupling is assumed to be attained through the exchange of impacts between the oscillators (clocks). This model presents the additional advantage of being independent of the physical nature of the oscillators, and thus can be used in other oscillator systems where synchronization and phase locking has been observed. The model presented starts from the Andronov model of the phase-space limit cycle of isolated pendulum clocks and assumes the exchange of single impacts (sound solitons, for this system) between the two clocks at a specific point of the limit cycle. Two coupling states are obtained, near phase and near phase opposition, the latter being stable. Our experimental data, obtained using a pair of similar pendulum clocks hanging from an aluminum rail fixed to a masonry wall, match the theoretical predictions and simulations.

Andronov model

The model for the isolated pendulum clock has been studied using models with viscous friction by physicists. However, Russian mathematicians lead by Andronov published a work where the stability of the model with dry friction is established (*Andronov clock*). The authors prove the existence and stability of the limit cycle.

Not so obvious: https://www.youtube.com/watch?v=SGgbRkix_hY

First explanation

- Huygens wrote about "sympathy of two clocks" in a letter to his father
- He also provided a qualitative explanation of this effect of *mutual synchronization*;
- he correctly understood that the conformity of the rhythms of two clocks had been caused by an *imperceptible motion of the beam*.

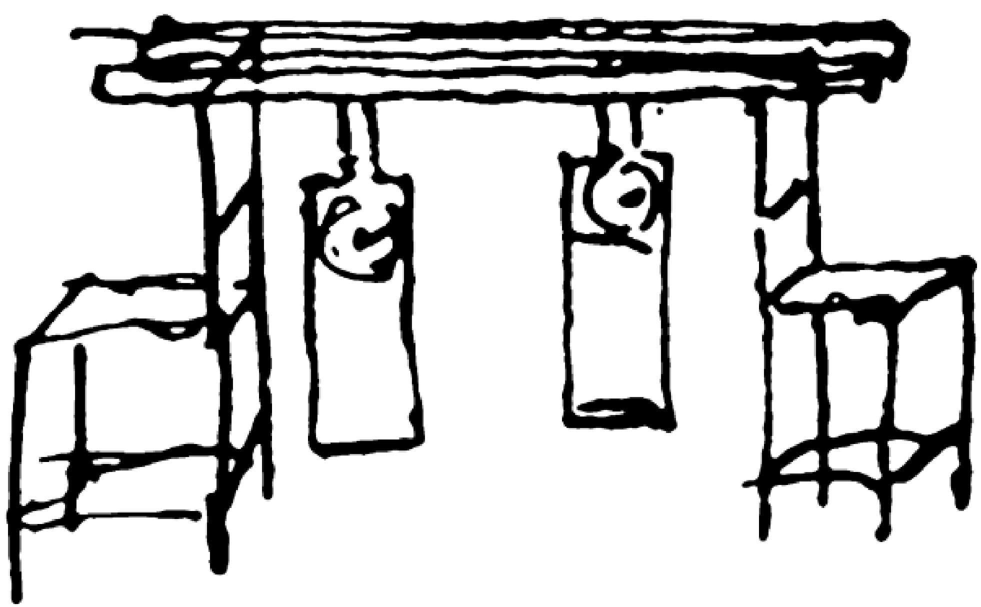
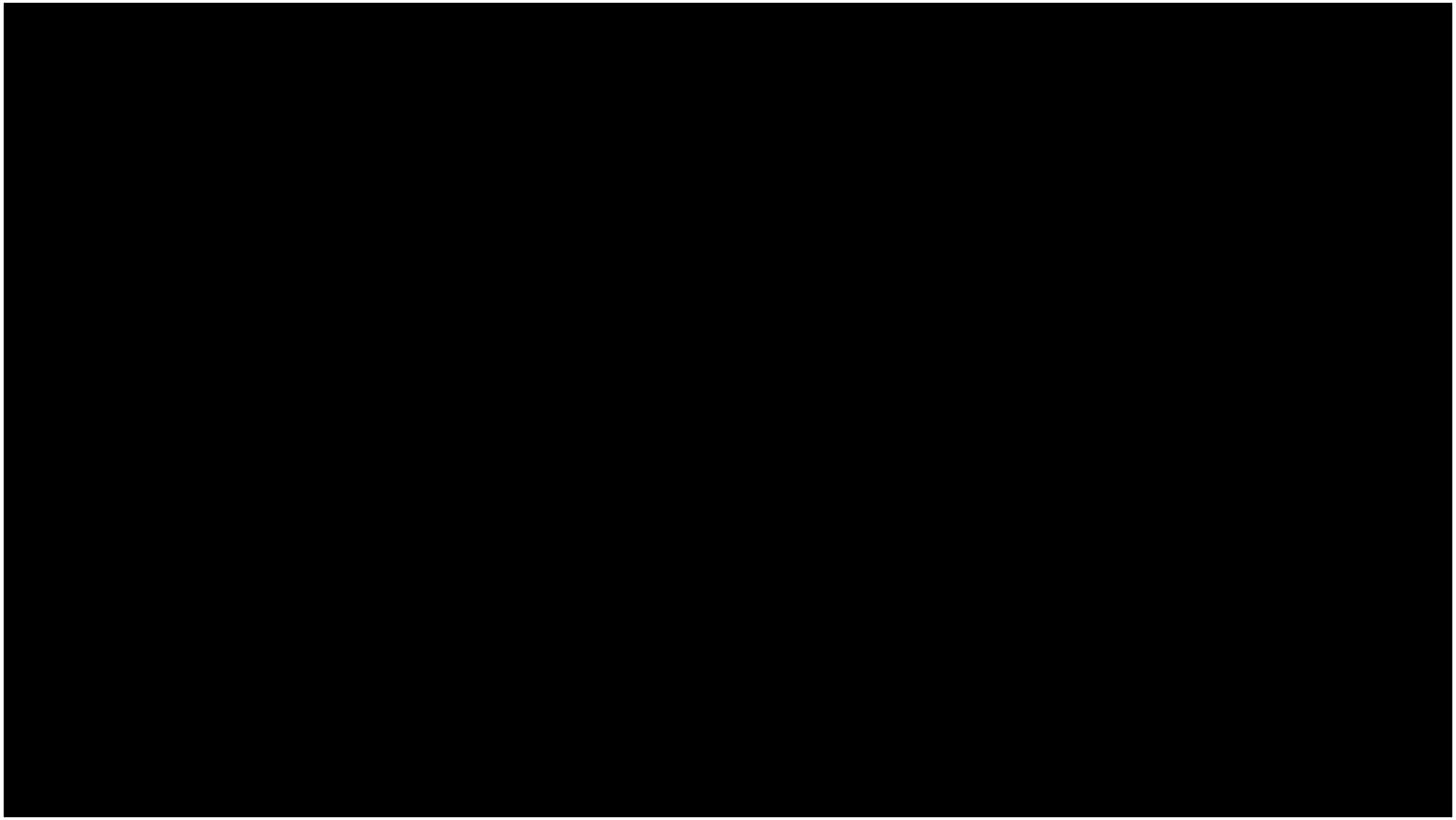


Figure 1.2. Original drawing of Christiaan Huygens illustrating his experiments with two pendulum clocks placed on a common support.

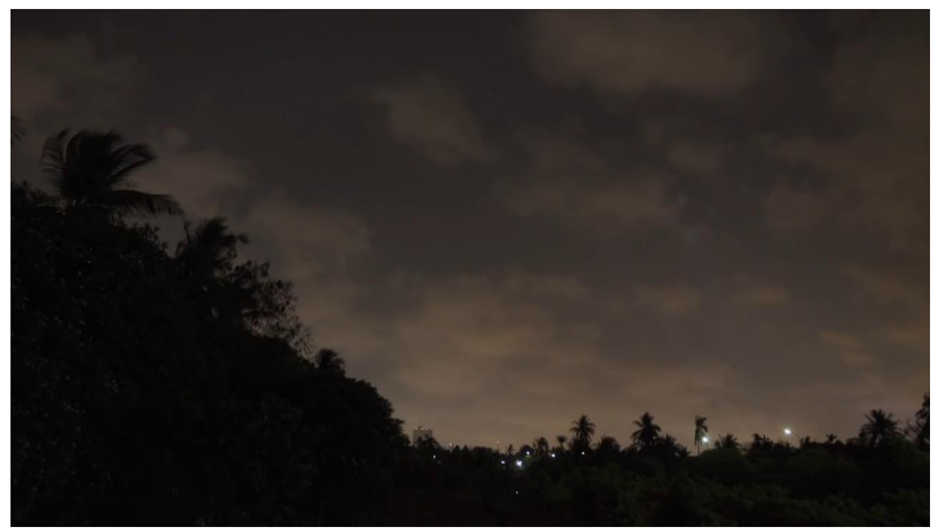
Oscillating metronomes – a demonstration



https://www.youtube.com/watch?v=bl2aYFv_978

- The burst into spontaneous applause
- Human physiology: walking, breathing
- Neuron network
- Pacemaker cells in the heart
- Chirping of crickets
- Fireflies
- Etc.

https://www.youtube.com/watch?v=ZGvtnE1Wy6U



https://www.youtube.com/watch?v=ZGvtnE1Wy6U

First models of biological oscillators

- Arthur Winfree, late 1960s
  - Ignored *all* biological differences and focused on the only common things: the ability to *send* and *receive signals*
  - Complication: both of these are often a function of phase
    - "Influence function" – what signal it sends
    - "Sensitivity function" – how an oscillator responds to the signals it receives
    - ➢ Oscillators can advance or delay, depending on where they are in their cycle when they receive a pulse. (Experiments show that most biological oscillators are like this)

❖ Assumptions:

- ❖ All the oscillators in a given population have the same influence and sensitivity function
- ❖ But the natural frequencies can vary, according to a bell shape
- ❖ Connectivity (the way the oscillators are connected)

Kuramoto model

- 1975
- assumptions:
  - the oscillators are identical or nearly identical (bell-shaped distribution of natural frequencies)
  - the interactions depend sinusoidally on the phase difference between each pair of objects.

- Later it has found widespread applications in other fields too (neuroscience, physical systems, etc.)

The Kuramoto model (KM)

- Continuous time and phase
- Consists of a population of *N* coupled oscillators
- Each tries to run independently at its own frequency, while the coupling tends to synchronize it to all the others
  - ϕ_i : the phase of oscillator *i* (in the sense of mod 2π)
  - t : time
  - T_i : periodic time
  - $\nu_i = \frac{1}{T_i}$: natural frequency
  - $\omega_i = \frac{2\pi}{T_i}$: natural angular frequency
- One oscillator (an oscillator without interaction):

$$\frac{d\phi}{dt} = \omega$$

The Kuramoto model in mean field approximation

- IN GENERAL: *N* coupled oscillators interacting with each others pairwise :

$$\frac{d\phi_i}{dt} = \omega_i + \sum_{j=0}^{N-1} \Gamma_{ij}(\phi_j - \phi_i), \qquad (i, j = 0,1, \dots, N-1)$$

- $\Gamma_{ij}(\Delta\phi)$: interaction, a function with 2π periodicity
- All the oscillators interact with each other the same way (this was the simplifying assumption of Kuramoto):

$$\Gamma_{ij}(\phi) = \frac{K}{N}\sin(\phi), \qquad (i, j = 0,1, \dots, N-1)$$

- K : strength of the coupling
- If $\mathrm{K} > 0 \ \rightarrow \Gamma$ minimizes the phase difference

The Kuramoto model in mean field approximation

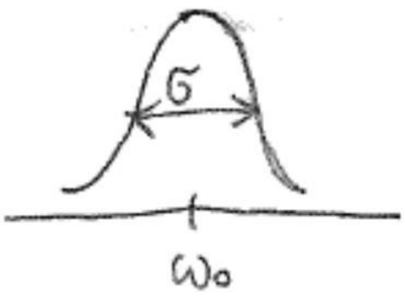
- The basic formula of the KM with MF approximation:

$$\frac{d\phi_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=0}^{N-1} \sin(\phi_j - \phi_i), \qquad (i, j = 0,1, \dots, N-1)$$

- How do such oscillators synchronize?
- The interplay between the coupling strength and the distribution of the natural frequencies determines how many oscillators are synchronized.
- How can we measure the level of synchronization?
  - Order parameter: An order parameter is a measure of the degree of order across the boundaries in a phase transition system; it normally ranges between zero in one phase and nonzero in the other.
- A trivial order parameter can be: $R = \frac{N_S}{N}$, where N_s is the number of synchronized units

Order parameter for the Kuramoto model

- The “Kuramoto order parameter” is more appropriate to monitor the transition towards synchronization)
- Let us assume that
  - the ω_i natural frequencies are taken from a Gaussian distribution $g(\omega)$
  - The expected value of the $g(\omega)$ density function is ω_0, with σ standard deviation



$$g(\omega) = \frac{1}{N}\sum_{i=0}^{N-1}\delta(\omega_i - \omega) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2}}$$

Defining the order parameter

- Parameter transformation:

$$\Psi_i \coloneqq \phi_i - \omega_0 t$$
$$\omega_i \leftarrow \omega_i - \omega_0$$

(ω_0 : average natural frequency)

- The Kuramoto formula is invariant to the above transformation:

$$\frac{d\psi_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=0}^{N-1} \sin(\psi_j - \psi_i) \;, (i, j = 0,1, \ldots, N-1)$$

- $\theta(t)$: the vectorial average of the (transformed) ψ_i unit vectors
- Now we can define the order parameter as next (as the *complex mean field* of the population):

$$z(t) \coloneqq Z(t)e^{i\theta(t)} = \frac{1}{N}\sum_{j=0}^{N-1} e^{i\psi_j(t)}$$

(here i is not the index of an oscillator, but $\sqrt{-1}$)

Defining the order parameter – cont.

$$\underbrace{z(t)}_{\text{Complex order param.}} := \underbrace{Z(t)}_{\text{Real part}} e^{i\theta(t)} = \frac{1}{N}\sum_{j=0}^{N-1} e^{i\psi_j(t)}$$

Complex order param.   Real part $\underbrace{\frac{1}{N} N \left| e^{i\psi_j(t)} \right|}_{=1}$

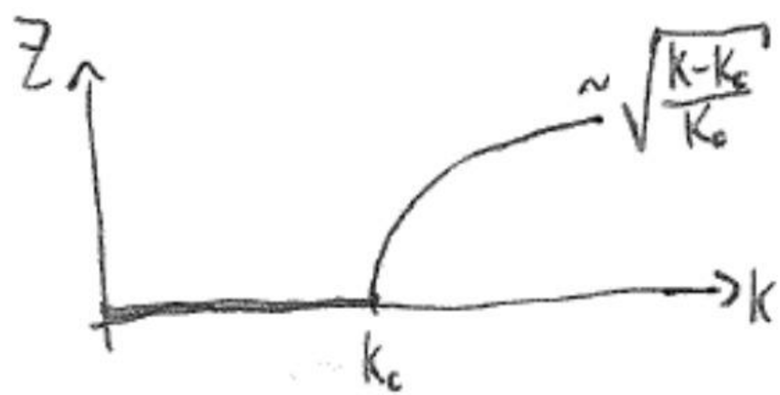
- real part of $z(t)$, $\rightarrow Z = |z|$
- the *order parameter* has the following properties:
  – Expresses the "closeness" of the ψ_i unitvectors
  – If $Z \approx 1 \rightarrow$ the ψ_i phases are close to each other
  – If $Z \approx 0 \rightarrow$ the ψ_i phases point in random direction

Bifurcation

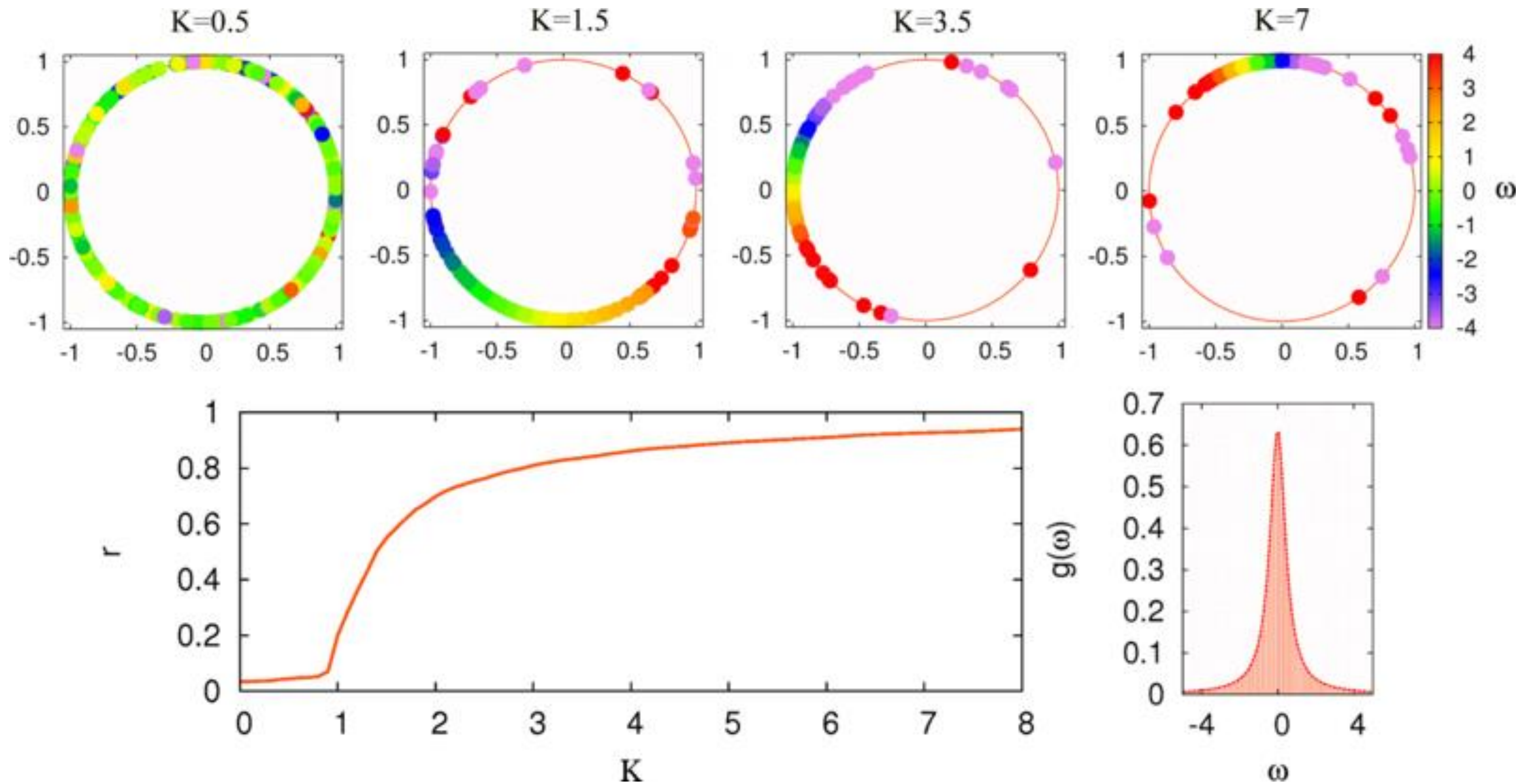
- In the uncoupled limit (K=0) each element i describes limit-cycle oscillations with characteristic frequency ω_i.
- Kuramoto showed that, by increasing the coupling K the system experiences a transition towards complete synchronization, i.e. , a dynamical state in which $\psi_i(t) = \psi_j(t)$ for $\forall i, j$ and $\forall t$.
- This transition shows up when the coupling strength exceeds a critical value whose exact value is

$$K_C = \frac{2}{\pi \cdot g(\omega_0)}$$

(ω_0 is the mean frequency of the $g(\omega)$ frequency distribution)



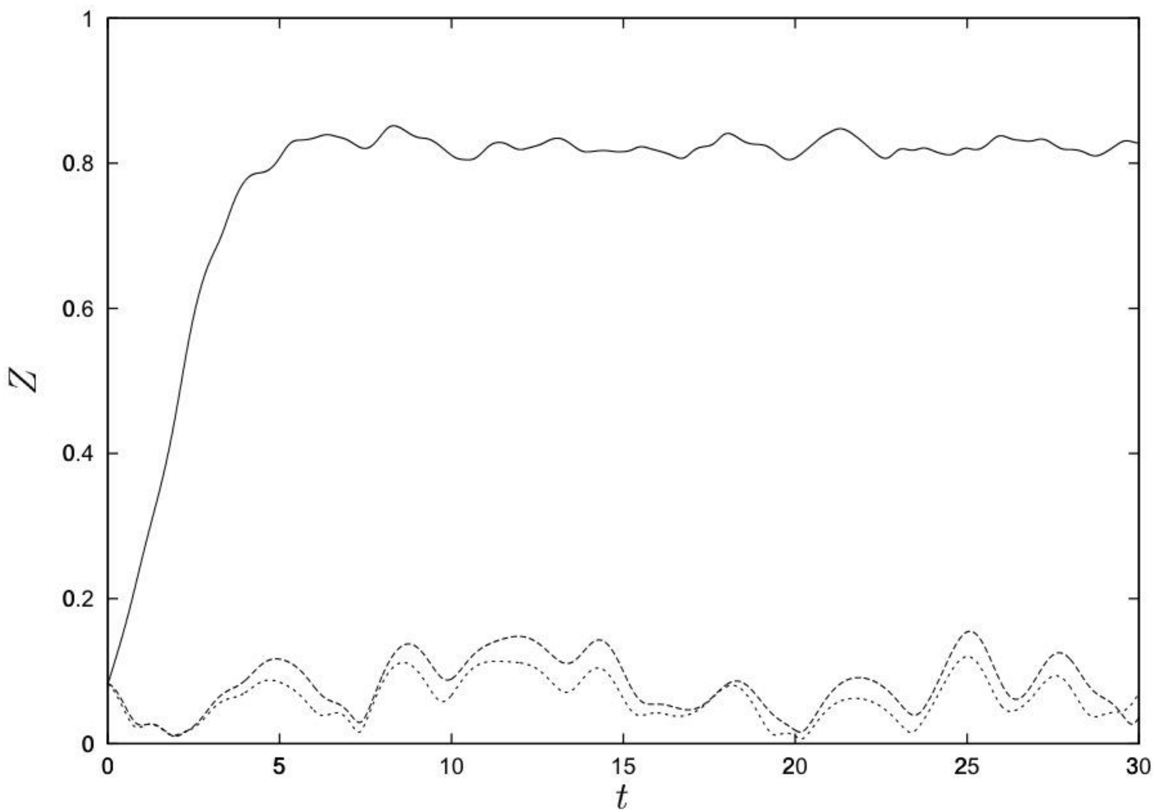
From: Mendoza et al., 2014, Synchronization in a semiclassical Kuramoto model.



Synchronization in the classical Kuramoto model. Each panel on the top shows the collection of oscillators situated in the unit circle (when each oscillator j is represented as $e^{i\psi_j(t)}$).
The color of each oscillator represents its natural frequency. From left to right we observe how oscillators start to concentrate as the coupling K increases. In the panels below we show the synchronization diagram, i.e. , the Kuramoto order parameter Z as a function of K . It is clear that $K_c = 1$.

From: Mendoza et al., 2014, Synchronization in a semiclassical Kuramoto model.

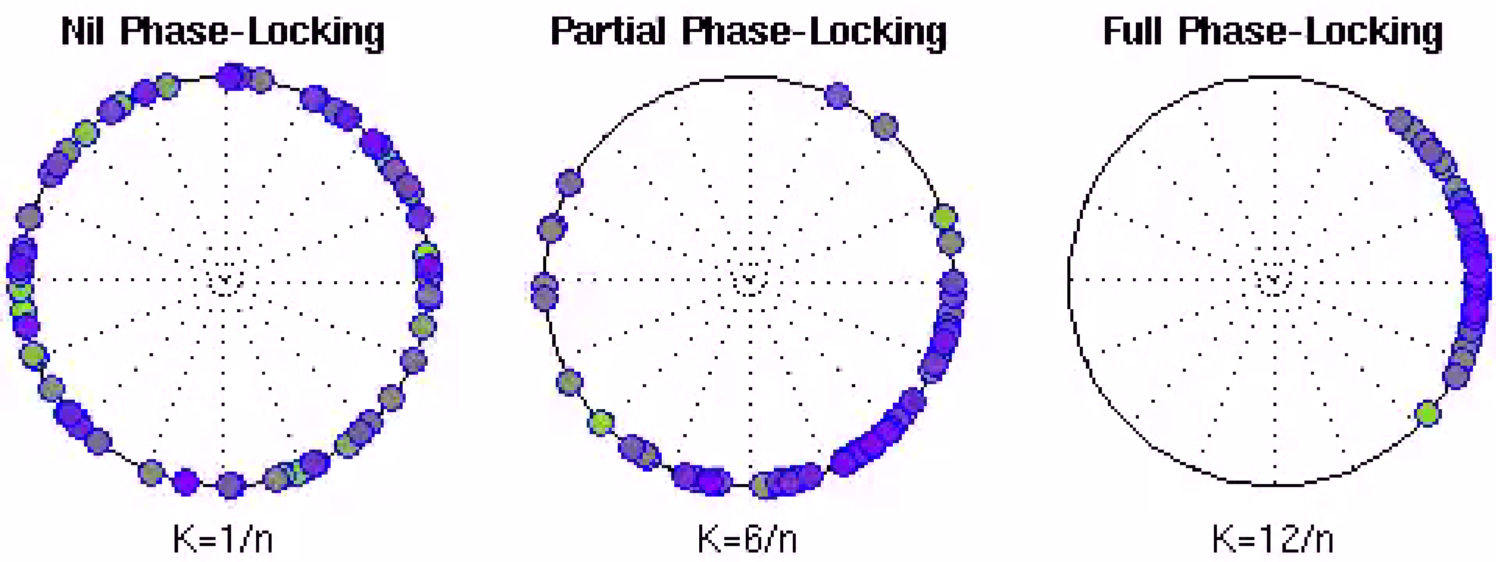
Simulation results



Z : order parameter
t : time
N = 200 coupled oscillators
σ = 1
K = 2.5 (top curve),
0.5 (middle curve)
0 (bottom curve)

→ K=0 and K=0.5 (weak coupling) results in similar order parameter

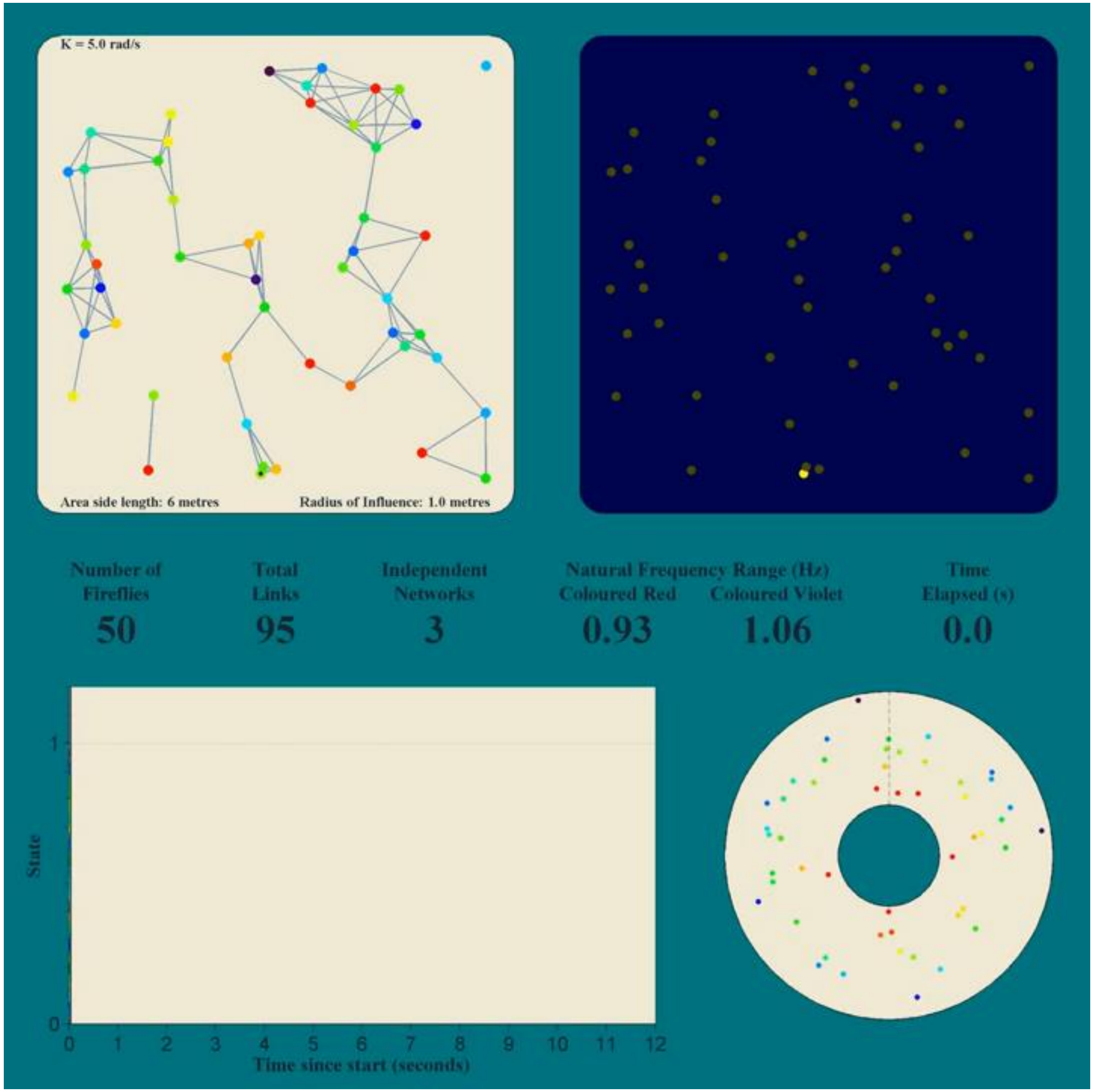
Phase-Coupled Oscillators



Nil, partial and full phase-locking behavior in a network of phase-coupled oscillators with all-to-all connectivity. The natural frequencies of the oscillators are normally distributed SD=±0.5Hz. The phase-locking behaviour is dictated by the strength of the global coupling constant K.

https://www.youtube.com/watch?v=9zrOoVlN8tg

Outlook: Kuramoto model on networks.



The all-to-all coupling considered originally by Kuramoto can be trivially generalized to any connectivity structures by introducing other coupling forms (via (weighted) adjacency matrices, graphs, etc.)

This allows for the study of the synchronization properties of a variety of real-world systems for which interactions between constituents are better described as a complex networks.

https://www.youtube.com/watch?v=hzRhdUkZc-s

Noise in the discrete Kuramoto model

- The KM with the above defined noise:

$$\frac{d\phi_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=0}^{N-1} \sin\left(\phi_j - \phi_i\right) + \xi_i$$

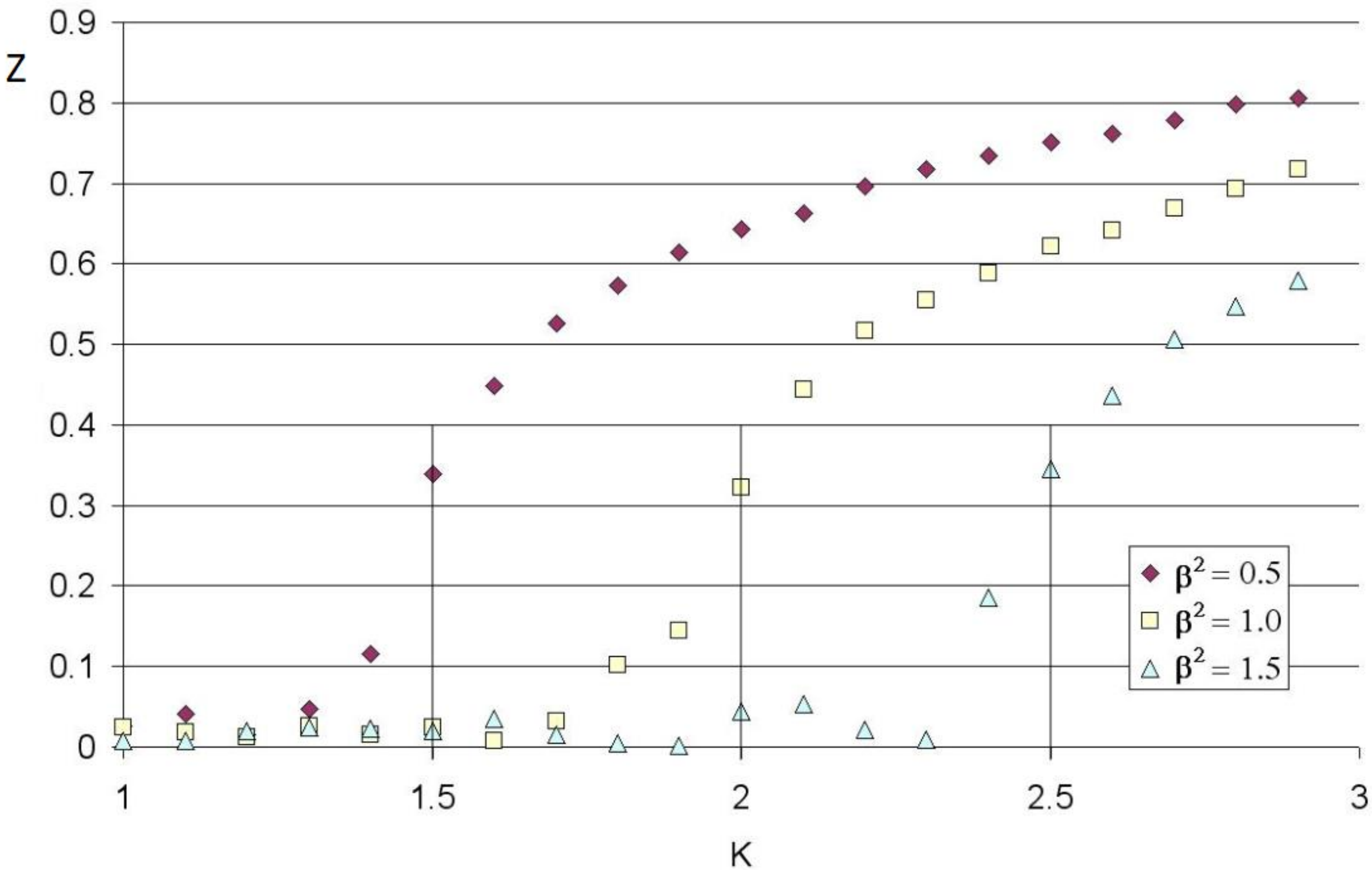
- Or in other form:

$$\frac{d\psi_i}{dt} = \omega_i + KZ\sin(\theta - \psi_i) + \xi_i$$

- ξ: a random value chosen from a normal (Gaussian) distribution of mean zero and width $\beta^2/\Delta t$, where
- β^2 defines the strength of the noise, and
- Δt is the time of the time-steps used in the simulations

Simulation results with white noise introduced to the discrete KM

N=5000



The dependency of the magnitude of the order parameter Z on the coupling K in presence of noise. β^2 sets the strength of the noise. From theoretical results K_C is predicted to occur at $\beta^2 + 1$, shown as three vertical lines at 1.5, 2.0, and 2.5.

From: Bryan C. Daniels: Synchronization of Globally Coupled Nonlinear Oscillators: the Rich Behavior of the Kuramoto Model, Fig 4.2.